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Inverse Generalized Gamma Distribution with it's properties

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Abstract:

In this paper, we introduce a new life time distribution . This distribution based on the reciprocal of Generalized Gamma (GG) random variable . This new distribution is called the Inverse Generalized Gamma (IGG) Distribution in which some of the inverse distributions are special cases. The important benefit of this distribution is ability to fit skewed data that cannot be fitted accurately by many other ungeneralized life time distributions. This distribution has many applications in pollution data ,engineering ,Biological fields and reliability. Some theoretical properties of the distribution has been studied such as: moments, mode, median and other properties.

It is concluded that the distribution is skew with heavy tail and the skewness increased when the shape parameters increased but the scale parameter has no effect on the skewness and kurtosis.

Key words: *Generalized Gamma distribution, Incomplete Gamma function, skewness and kurtosis.*

توزيع معكوس كاما المعمم مع بعض خصائصه

المستخلص :

قدمنا في هذا البحث توزيع احتمالي جديد لأزمنة البقاء. وقد استند ايجاد هذا التوزيع على مقلوب متغير عشوائي ذو توزيع كاما المعمم. وقد سمي التوزيع الجديد بتوزيع معكوس كاما المعمم(IGG) الذي تكون بعض التوزيعات الاحتمالية المعكوسة حالات خاصة منه. ان اهمية هذا التوزيع هي قدرته على ملائمة البيانات الملتوية والتي من غير الممكن ملائمتها بدقة باستخدام العديد من توزيعات ازمنة البقاء غير المعممة.

هذا التوزيع له العديد من التطبيقات في بيانات التلوث وفي المجالات الهندسية والحياتية والموثوقية ، درست بعض الخصائص النظرية للتوزيع كالعزوم، المنوال ، الوسيط، مع خصائص أخرى. وقد تبين ان التوزيع ملتوٍ وذو ذيل ثقيل ويزداد الالتواء بزيادة قيمتي معلمتي الشكل ، لكن معلمة القياس ليس لها تأثير على الالتواء والتفلطح .

الكلمات المفتاحية : *توزيع كاما المعمم ، دالة كاما غير الكاملة، الالتواء والتفلطح.*

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(1) Introduction:

The continuous distributions can be generalized by elevating the c.d.f. G(x) of any continuous random variable to a power parameter where this parameter defined on positive real number.

This approach was firstly used by Stacy (1962) who generalized the two parameters Gamma Distribution to three parameters (GG) Distribution. Al-Saqabi et.al (2007) generalized the Inverse Gaussian Distribution. Khodabin and Ahmedabadi (2010) gave some properties of generalized gamma distribution and estimated some of its parameters by moment method. Felipe et, al (2011) gave the generalization of Inverse Weibull distribution. El-Gohery, et, al (2013) took a generalization of Gompertz distribution . . Gauhar (2015) put a Generalized Chi- square distribution by using the K-gamma function Rabbit and Madi (2017) introduced the Generalized Raileigh distribution

This paper organized as: In section two , a (IGG) density function , and some special cases of the distribution. Some statistical properties of (IGG) contained in section 3.

(2) Inverse Generalized Gamma distribution (IGG):

The p.d.f. of a random variable X which follows a GG($\propto,\beta,\frac{1}{\lambda}$) defined as : (Khodabin and Ahmedabadi(2010))

$$f(X,\propto,\beta,\lambda)=\frac{\beta\, X^{\alpha\beta-1}\, e^{-\left(\frac{X}{\lambda}\right)^{\beta}}}{\lambda^{\alpha\beta}\,\Gamma\alpha}\qquad X,\alpha,\beta,\lambda>0 \qquad \text{... (1)}$$

Where α,β be the shape parameters , and λ be the scale parameter .

The one-to-one transformation $Y=\frac{1}{X}$, $Y>0$,the p.d.f. of Y has the following form

$$f(Y,\propto,\beta,\lambda)=\frac{\beta\, Y^{-(\alpha\beta+1)}\, e^{-\left(\frac{1}{\lambda Y}\right)^{\beta}}}{\lambda^{\alpha\beta}\,\Gamma\alpha}\qquad Y,\alpha,\beta,\lambda>0 \qquad \text{... (2)}$$

Where $\Gamma\alpha$ be the complete gamma function .

We call the p.d.f. defined in eq(2) as the Inverse Generalized Gamma distribution (IGG) , denoted by $Y\sim IGG\left(\alpha,\beta,\frac{1}{\lambda}\right)$, α,β represent shape parameters and λ be the scale parameter .

Figure (1) below represents the curve of the p.d.f. defined in eq(2) at fixed scale parameter and different values of shape parameters.

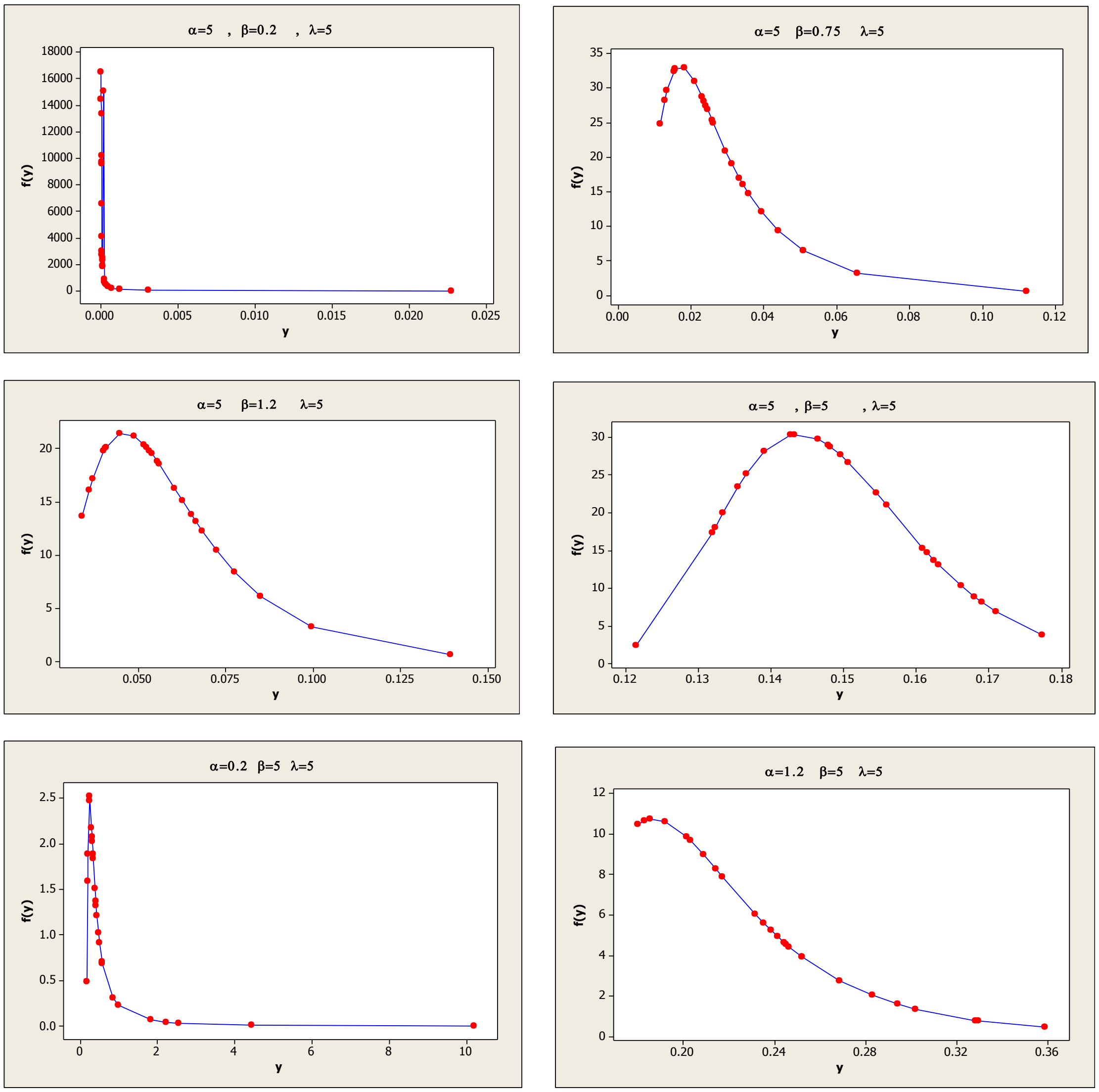

Figure (1):The p.d.f. curve of IGG distribution at $\lambda = 5$and different values of α and β

It is seen from the figure(1) that the curves are positive skew with heavy tails. These tails become thicker when the shape parameters are equal and the tails become longer when the shape parameter ($\boldsymbol{\beta}$) increases**.**

If $\beta = 1$, IGG reduces to Inverse Gamma (IG) distribution .When $\alpha = 1$, the distribution reduces to Inverse Weibull $\left(\beta, \frac{1}{\lambda}\right)$, But when $\left(\lambda = 2\,, \beta = 1,\ \alpha = \frac{k}{2}\right)$ where k is a positive integer , the distribution reduces to Inverse Chie-square with k degrees of freedom . While $(\alpha = 1, \beta = 2, \lambda^2 = 2\sigma^2\ ,\ \sigma^2 > 0\ )$, it reduces to Inverse Rayleigh with parameter σ^2 .

The cumulative distribution function of Y is :

$$F(y) = \frac{1}{\Gamma\alpha}\,\Gamma\left(\alpha, \left(\frac{1}{\lambda y}\right)^{\beta}\right) \qquad \ldots (3)$$

Where $\lambda > 0$ is a scale parameter and $\alpha > 0$ is the shape parameter and

$\Gamma(a, b) = \int_b^{\infty} u^{a-1} e^{-u}\, du$ is an upper incomplete gamma

The reliability function is

$$R(y) = 1 - F(y) = \frac{\gamma\left(\alpha, \left(\frac{1}{\lambda y}\right)^{\beta}\right)}{\Gamma\alpha} \qquad \ldots (4)$$

Where $\gamma(a, b) = \int_0^{b} u^{a-1} e^{-u}\, du$ is a lower incomplete gamma function and the hazard rate function

$$h(y) = \frac{f(y)}{R(y)} = \frac{\beta\, y^{-(\alpha\beta+1)}\, e^{-\left(\frac{1}{\lambda y}\right)^{\beta}}}{\lambda^{\alpha\beta}\, \gamma\left(\alpha, \left(\frac{1}{\lambda y}\right)^{\beta}\right)} \qquad \ldots (5)$$

For $\lambda = 5$ and different values for α, β the $f(y)$ and is plotted in figures (1).

(3)Properties of the distribution :

In this section , some properties of the distribution will be discussed as follows:

(3-1) Moments

The r-th moment around zero of $Y \sim IGG\left(\alpha, \beta, \frac{1}{\lambda}\right)$ can be obtained as:

$$EY^r = \frac{\beta}{\lambda^{\alpha\beta}\, \Gamma\alpha} \int_0^{\infty} Y^r\, Y^{-(\alpha\beta+1)}\, e^{-\left(\frac{1}{\lambda Y}\right)^{\beta}}\, dY \qquad \ldots (6)$$

The integral in eq(6) can be reduced to the integral of the kernel of inverse Gamma distribution by defining a one-to-one transformation $Z = (\lambda Y)^{\beta}$, $Z > 0$, therefore eq(6) reduces to

$$EY^r = \frac{1}{\lambda^r\, \Gamma\alpha} \int_0^{\infty} Z^{-\left(\alpha - \frac{r}{\beta} + 1\right)}\, e^{-\left(\frac{1}{Z}\right)^{\beta}}\, dZ$$

$$\Rightarrow EY^r = \frac{\Gamma\left(\alpha - \frac{r}{\beta}\right)}{\lambda^r\, \Gamma\alpha} \quad , \quad for \quad \alpha > \frac{r}{\beta} \qquad \ldots (7)$$

The first and second moment Y are

$$\left.\begin{aligned} EY &= \frac{\Gamma\left(\alpha-\frac{1}{\beta}\right)}{\lambda\,\Gamma\alpha} \quad , \quad for \quad \alpha > \frac{1}{\beta} \\ EY^2 &= \frac{\Gamma\left(\alpha-\frac{2}{\beta}\right)}{\lambda^2\,\Gamma\alpha} \quad , \quad for \quad \alpha > \frac{2}{\beta} \end{aligned}\right\} \quad \ldots (8)$$

Thus $Var(Y) = \frac{1}{\lambda^2}\left[\frac{\Gamma\left(\alpha-\frac{2}{\beta}\right)}{\Gamma\alpha} - \left(\frac{\Gamma\left(\alpha-\frac{1}{\beta}\right)}{\Gamma\alpha}\right)^2\right] \quad , for \quad \alpha > \frac{2}{\beta}$ ... (9)

The characteristic function (c.f.) of Y is :

$$\phi_y(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} EY^r \quad \ldots (10)$$

Substituting eq(7) into 10 , the c.f. of Y will be

$$\phi_y(t) = \sum_{r=0}^{\infty} \left(\frac{it}{\lambda}\right)^r \frac{\Gamma\left(\alpha-\frac{r}{\beta}\right)}{r!\ \Gamma\alpha} \quad \ldots (11)$$

(3-2) Median and Mode :

The median of Y is a solution of the $\frac{\Gamma\left(\alpha,\left(\frac{1}{\lambda Y}\right)^{\beta}\right)}{\Gamma\alpha} = \frac{1}{2}$ with respect to Y.

And the mode is a solution of the following equation

$$(\alpha\beta + 1)\lambda^{\beta}\, Y^{\beta} - \beta = 0 \quad \ldots (12)$$

The mode be :-

$$Mo = \frac{1}{\lambda}\left(\frac{\beta}{\alpha\beta + 1}\right)^{\frac{1}{\beta}} \quad \ldots (13)$$

The mean and mode of the distribution evaluated at different values of parameters.

From table (1) below it is shown that the mean and mode decreases when the shape parameters are fixed and scale parameter increases. Also when the shape parameter (α) and scale parameter are fixed and (β) increases the mean and mode increased, but when (β, λ) are fixed and (α) increased, the mean and mode decreased. At all values of the parameters, the mean is greater than the mode. This is an indication that the distribution has a positive skew.

Table (1): The mean and mode of IGG distribution at different values of shape and scale parameters

λ		2		4	
α	β	mean	mode	Mean	mode
1.5	3	0.523410	0.408529	0.261705	0.204264
	4.25	0.509741	0.439184	0.254870	0.219592
	5	0.506344	0.449656	0.253172	0.224828
	6.25	0.503379	0.461055	0.251689	0.230527
2.5	3	0.407096	0.353349	0.203548	0.176675
	4.25	0.429781	0.394590	0.214891	0.197295
	5	0.438831	0.409918	0.219416	0.204959
	6.25	0.449685	0.427552	0.224842	0.213776
3	3	0.376144	0.334716	0.188072	0.167358
	4.25	0.406981	0.379307	0.203490	0.189653
	5	0.419123	0.396223	0.209561	0.198112
	6.25	0.433602	0.415930	0.216801	0.207965
4	3	0.334350	0.306687	0.167175	0.153344
	4.25	0.375061	0.356015	0.187530	0.178007
	5	0.391181	0.375250	0.195591	0.187625
	6.25	0.410476	0.398029	0.205238	0.199015

The skewnees and kurtosis of IGG distribution was evaluated at different values of parameters which they shown in table (2).

From table (2) below it is seen the shape parameters have reverse effect on skewness and kurtosis.

Table(2):Skewness and kurtosis of IGG distribution at different values of parameters

α	β	λ			
		2		4	
		skewness	**Kurtosis**	**skewness**	**Kurtosis**
1.5	**3**	81.6584	0.544786	81.6584	0.544786
	4.25	15.0755	0.530259	15.0755	0.530259
	5	10.2400	0.517762	10.2400	0.517762
	6.25	6.918611	0.499182	6.918611	0.499182
2.5	**3**	8.432404	0.508423	8.432404	0.508423
	4.25	4.505593	0.463815	4.505593	0.463815
	5	3.620759	0.444409	3.620759	0.444409
	6.25	2.821866	0.420374	2.821866	0.420374
3	**3**	5.598654	0.483360	5.598654	0.483360
	4.25	3.278615	0.435260	3.278615	0.435260
	5	2.698537	0.415377	2.698537	0.415377

	6.25	2.152816	0.391281	2.152816	0.391281
4	**3**	3.305776	0.439223	3.305776	0.439223
	4.25	2.105860	0.389626	2.105860	0.389626
	5	1.775488	0.370076	1.775488	0.370076
	6.25	1.451457	0.346897	1.451457	0.346897

(4)Conclusions:

The IGG distribution was found by the reciprocal of GG variable . It is found that many life time distributions are special cases of IGG . This distribution is a positive skew with heavy tails and degree of skewness and kurtosis decreased when the shape parameters inceased.

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