Point and Interval Estimation of Stress-Strength Model for Exponentiated Inverse Rayleigh distribution

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Abstract

This paper deals with finding a formula for the stress-strength reliability function $P(T < X < Z)$ for complete data when the strength $(X)$ falls between the stress $(T)$ and the stress $(Z)$, where $X,T,Z$ are independent random variables and follow the Exponentiated Inverse Rayleigh Distribution with unknown shape parameters and a common known scale parameter, and estimate this formula with the Maximum Likelihood Estimate method (MLE) and the Bayesian method using Non-informative priors and informative priors under Weighted Square Error Loss Function (WSELF) and Simulation study is used to determine the best estimator, the results showed that Bayesian estimation using informative priors based on Weighted Square Error Loss Function is the best estimator for the equal sizes, and Bayesian estimation using Non-informative priors based on Weighted Square Error Loss Function is the best estimator when the size of the stress sample $(Z)$ larger than the size of $(X,T)$, and Maximum Likelihood Estimator is the best estimator for the rest sizes.

Keywords: Point and Interval Estimation, Stress-Strength Model, Weighted Square Error Loss Function, Exponentiated Inverse Rayleigh Distribution.

Introduction

Exponentiated Inverse Rayleigh distribution (EIR) is a life time distribution used in reliability estimation and statistical quality control techniques. It's a generalization of inverse Rayleigh distribution that developed by Nadarajah and Kotz (1). They suggested a method of generating new exponential type distribution by using reliability function:

$$F(x) = 1 - \{R(x)\}^\alpha$$

Where $R(x)$ is the reliability function of Inverse Rayleigh distribution.

The C.D.F of Exponentiated Inverse Rayleigh distribution is:

$$F(x) = 1 - \left(1 - e^{-\left(\frac{x}{\lambda}\right)^2}\right)^\alpha, \quad \alpha > 0$$

And the P.D.F of EIR distribution is:

$$f(x) = \frac{2\alpha x^2}{\lambda^2} e^{-\left(\frac{x}{\lambda}\right)^2} \left(1 - e^{-\left(\frac{x}{\lambda}\right)^2}\right)^{\alpha-1}, \quad x > 0, \alpha > 0, \lambda > 0$$

where $\lambda$ indicates the scale parameter and $\alpha$ indicates the shape parameter.

Note that When $\alpha = 1$ the Exponentiated Inverse Rayleigh distribution (EIR) distribution turns into Inverse Rayleigh (IR) distribution.
As for the reliability of the stress-strength (S-S.R.), it has two types (classical and modern) stress-strength, the classical stress-strength explained the life of the component and describe the ability (strength (x)) of the component to still functional when it subject to random stress (T), and interest to estimate the probability of the component's strength (X) exceed the stress (T); P(T < X).

And the component either fail or the system containing the component might malfunction when (T ≥ X). The second type is P(T<X<Z); which the current study concerned with evaluating and estimating, P(T<X<Z) represent that the strength of the component (X) should not be only greater than the component's stress(T) but also should be smaller than the other component's stress (Z).

For example, blood pressure which has two limits (systolic and diastolic) and the person’s blood pressure should be between these limits (2).

In the past 45 years; a case of stress-strength reliability P(T<X<Z) considered when the cumulative functions of T and Z are known and pdf of X is unknown but its observation is available (3). The reliability estimated where X, T and Z are independent and follow a Weibull distribution with different unknown scale parameters and commonly known shape parameter, in presence of k outliers in the strength X, the moment estimator and maximum likelihood (MLE) estimators and mixture estimators of the reliability are derived (4). Then the reliability R = P(X<T<Z) was estimated using Monte-Carlo simulation (MCS) for n-standby system when both of stress and strength follows a particular continuous distribution (5). And the stress-strength reliability estimated using Maximum Likelihood, Method of Moment, Least Square Method, and Weighted Least Square Method when X, T, Z are followed New Weibull-Pareto Distribution with unknown shape parameter (6).

**Reliability formula**

Deriving The formula of the reliability of stress-strength function P(T<X<Z) under complete data for a component's strength (X) that falls in between the stresses T and Z respectively, will be as follows (7):

\[ R = P(T < X < Z) = \int_0^\infty P(T < X < Z)f(x)dx \]

where \(X, T, Z\) are all independent

\[ R = \int_0^\infty H_1(x) \left(1 - G_2(x)\right) f(x)dx \]

where \(H(x) = 1 - \left(1 - e^{-\left(\frac{x}{\alpha_1}\right)^{\alpha_1}}\right)\)

\(G_2(x) = 1 - \left(1 - e^{-\left(\frac{x}{\alpha_2}\right)^{\alpha_2}}\right)\)

\[ A_1 = \int_0^\infty f(x)dx - \int_0^\infty \left(1 - e^{-\left(\frac{x}{\alpha_1}\right)^{\alpha_1}}\right) f(x)dx \]

\[ = 1 - \frac{a}{a+\alpha} \left(1 - \frac{a}{a+\alpha}\right) \]

\[ A_2 = \int_0^\infty G_2(x) f(x)dx \]

\[ A_2 = \int_0^\infty \left[1 - \left(1 - e^{-\left(\frac{x}{\alpha_2}\right)^{\alpha_2}}\right)\right] \left[1 - \left(1 - e^{-\left(\frac{x}{\alpha_1}\right)^{\alpha_1}}\right)\right] \frac{a_1}{a_1+a_2} e^{-\left(\frac{x}{\alpha_2}\right)^{\alpha_2}} \left(1 - e^{-\left(\frac{x}{\alpha_1}\right)^{\alpha_1}}\right)^{a_1-1} dx \]

\[ A_2 = 1 - \frac{\alpha}{\alpha + \alpha_1} - \frac{\alpha}{\alpha + \alpha_2} + \frac{\alpha}{\alpha + \alpha_1 + \alpha_2} \]

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Substituting the result of $A_1$ and $A_2$ in R to get the formula of R

$$R = \frac{\alpha_1}{(\alpha + \alpha_2)(\alpha + \alpha_1 + \alpha_2)} , \quad 0 < R < 1$$

**Point Estimation**

Point estimation is a process of finding an approximate value of unknown parameters from statistics taken from one or several samples of the population. This section shall discuss two types of point estimation (maximum likelihood estimation, Bayesian estimation).

**Maximum likelihood estimation**

Let $\{x_i, i = 1, 2, ..., n\}$ be a sample of random observations of strength taken from EIR $(\alpha, \lambda)$ with known scale parameter $\lambda > 0$ and unknown shape parameter $\alpha > 0$, then the likelihood function of the sample $x$ is given by:

$$L(x|\alpha) = B \alpha^n \prod_{i=1}^{n} \left(1 - e^{-\left(\frac{x_i}{\alpha}\right)^2}\right)^{\alpha}$$

Where: $$B = 2^{n-1} \lambda^{2n} \left(\prod_{i=1}^{n} x_i^{-3}\right) e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\alpha}\right)^2} \prod_{i=1}^{n} \left(1 - e^{-\left(\frac{x_i}{\alpha}\right)^2}\right)^{-1}$$

And let $\{t_j, j = 1, 2, ..., m\}$ and $\{z_k, k = 1, 2, ..., w\}$ be samples of random stresses observations taken from EIR$(\alpha_1, \lambda), \text{EIR}(\alpha_2, \lambda)$ respectively, that their scale parameter $\lambda > 0$ is known and equal and shape parameters $\alpha_1, \alpha_2 > 0$ are unknown. $t_j$ and $z_k$ are independent from each other and from $x_i$, then the likelihood functions of the samples $t, z$ are given by:

$$L(t|\alpha_1) = B_1 \alpha_1^m \prod_{j=1}^{m} \left(1 - e^{-\left(\frac{t_j}{\alpha_1}\right)^2}\right)^{\alpha_1}$$

Where: $$B_1 = 2^{m-1} \lambda^{2m} \left(\prod_{j=1}^{m} t_j^{-3}\right) e^{-\sum_{j=1}^{m} \left(\frac{t_j}{\alpha_1}\right)^2} \prod_{j=1}^{m} \left(1 - e^{-\left(\frac{t_j}{\alpha_1}\right)^2}\right)^{-1}$$

$$L(z|\alpha_2) = B_2 \alpha_2^w \prod_{k=1}^{w} \left(1 - e^{-\left(\frac{z_k}{\alpha_2}\right)^2}\right)^{\alpha_2}$$

Where: $$B_2 = 2^{w-1} \lambda^{2w} \left(\prod_{k=1}^{w} z_k^{-3}\right) e^{-\sum_{k=1}^{w} \left(\frac{z_k}{\alpha_2}\right)^2} \prod_{k=1}^{w} \left(1 - e^{-\left(\frac{z_k}{\alpha_2}\right)^2}\right)^{-1}$$

The maximum likelihood estimators of the parameters $(\alpha, \alpha_1, \alpha_2)$:

$$\hat{\alpha}_{mle} = \frac{-n}{\sum_{i=1}^{n} \ln \left(1 - e^{-\left(\frac{x_i}{\alpha}\right)^2}\right)} , \quad \hat{\alpha}_{1mle} = \frac{-m}{\sum_{j=1}^{m} \ln \left(1 - e^{-\left(\frac{t_j}{\alpha_1}\right)^2}\right)} , \quad \hat{\alpha}_{2mle} = \frac{-w}{\sum_{k=1}^{w} \ln \left(1 - e^{-\left(\frac{z_k}{\alpha_2}\right)^2}\right)}$$

$$\hat{\alpha}_{mle} > 0, \hat{\alpha}_{1mle} > 0, \hat{\alpha}_{2mle} > 0$$

The MLE for the (S-S.R.) can be found by applying the invariance property on R for the MLE of $\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2$

$$\hat{R}_{mle} = \frac{\hat{\alpha}_{mle} \hat{\alpha}_{1mle}}{\hat{\alpha}_{mle} + \hat{\alpha}_{2mle} + \hat{\alpha}_{1mle} + \hat{\alpha}_{2mle}} , \quad 0 < \hat{R}_{mle} < 1$$
Bayesian estimation

This part estimates the stress-strength reliability using Bayesian estimation method and under consideration that it performed for complete data by using informative and non-informative priors based on Weighted Squared Error loss function (W.S.E.L.F)

A. Bayesian estimation using Non-informative Jeffrey's prior based on Weighted Squared Error loss function

The non-informative Jeffrey's prior for the shape parameter $\alpha$ is (8):

$$P(\alpha) \propto \sqrt{I_X(\alpha)}$$

where $I_X(\alpha)$ is the Fisher information for the parameter $\alpha$

The non-informative prior for $(\alpha, \alpha_1, \alpha_2)$:  $P(\alpha) \propto \frac{1}{\alpha}, P(\alpha_1) \propto \frac{1}{\alpha_1}, P(\alpha_2) \propto \frac{1}{\alpha_2}$

The posterior distribution for $\alpha$ is

$$P_{BNW}(\alpha | \mathbf{x}) \propto \alpha^{n-1}e^{-\frac{\sum_{i=1}^{n} \ln \left(1 - e^{-\left(\frac{\mathbf{x}_i}{\alpha} \right)^2} \right)}{\alpha}}$$

Which is the kernel of gamma distribution $G(n, D)$ where $D = -\sum_{i=1}^{n} \ln \left(1 - e^{-\left(\frac{\mathbf{x}_i}{\alpha} \right)^2} \right)$

Then the complete posterior distribution Of $P_{BNW}(\alpha | \mathbf{x})$ is:

$$P_{BNW}(\alpha | \mathbf{x}, \mathbf{t}) = \frac{d^n}{\Gamma_n \Gamma_m \Gamma_w} \alpha^{-m-1}e^{-\alpha \mathbf{D}_1}$$

Similarly, the posterior distribution for $\alpha_1$ and $\alpha_2$ are

$$P_{BNW}(\alpha_1 | \mathbf{x}, \mathbf{t}) = \frac{d^n}{\Gamma_n \Gamma_m \Gamma_w} \alpha_1^{-m-1} \alpha_1^{-w-1} e^{-\alpha_1 \mathbf{D}_1}$$

$$P_{BNW}(\alpha_2 | \mathbf{x}, \mathbf{t}) = \frac{d^n}{\Gamma_n \Gamma_m \Gamma_w} \alpha_2^{-w-1} e^{-\alpha_2 \mathbf{D}_2}$$

Since $\mathbf{x}$ and $\mathbf{t}$ are independent The joint posterior can be found as follows:

$$P_{BNW}(\alpha, \alpha_1, \alpha_2 | \mathbf{x}, \mathbf{t}, \mathbf{z}) = \frac{d^n}{\Gamma_n \Gamma_m \Gamma_w} \alpha^{-m-1} \alpha^1^{-m-1} \alpha_2^{-w-1} e^{-\alpha \mathbf{D}_1} e^{-\alpha_1 \mathbf{D}_1} e^{-\alpha_2 \mathbf{D}_2}$$

The Weighted Squared Error loss function (9) takes the following form:

$$L(R, \hat{R}) = \left( \frac{\hat{R} - R}{R} \right)^2$$

To find the Bayesian estimation $(\hat{R})$ for (S-S.R.) based on Weighted Squared Error loss function we solved the following equation:

$$\frac{\partial E[L(R, \hat{R})]}{\partial \hat{R}} = 0 \quad , \quad \hat{R}_W = \frac{1}{E(\alpha^{-1} | \mathbf{x}, \mathbf{t}, \mathbf{z})}$$

The expectation in the denominator using Non-informative Jeffrey's prior based on Weighted Squared Error loss function is:

$$E_{BNW}(R^{-1} | \mathbf{x}, \mathbf{t}, \mathbf{z}) = \int_0^\alpha \int_0^\alpha \int_0^\alpha R^{-1} P_{BNW}(\alpha, \alpha_1, \alpha_2 | \mathbf{x}, \mathbf{t}, \mathbf{z}) \, d\alpha \, d\alpha_1 \, d\alpha_2$$

$$= \int_0^\alpha \int_0^\alpha \int_0^\alpha \frac{d^n}{\Gamma_n \Gamma_m \Gamma_w} \alpha^{-m-1} \alpha_1^{-m-1} \alpha_2^{-w-1} e^{-\alpha \mathbf{D}_1} e^{-\alpha_1 \mathbf{D}_1} e^{-\alpha_2 \mathbf{D}_2} \, d\alpha \, d\alpha_1 \, d\alpha_2$$

$$E_{BNW}(R^{-1} | \mathbf{x}, \mathbf{t}, \mathbf{z}) = \frac{d^n}{\Gamma_n \Gamma_m \Gamma_w} \int_0^\alpha \int_0^\alpha \int_0^\alpha \alpha^{-m-1} \alpha_1^{-m-1} \alpha_2^{-w-1} e^{-\alpha \mathbf{D}_1} e^{-\alpha_1 \mathbf{D}_1} e^{-\alpha_2 \mathbf{D}_2} \, d\alpha \, d\alpha_1 \, d\alpha_2$$

By solving the integrations which is kernels of gamma distribution
\[ E_{BNW}(R^{-1|x, t, Z}) = 1 + 2 \frac{(w)D_1}{(m-1)D_2} + \frac{(w+1)D}{(n-1)(m-1)D_2} + \frac{(n)D}{(m-1)D} \]

Substituting equation above in \( \hat{R}_w \) to get the Bayesian estimation using non-informative prior based on Weighted Square Error Loss Function:

\[ \hat{R}_{BNW} = \left[ 1 + 2 \frac{w D_1}{(m-1)D_2} + \frac{w D}{(n-1)(m-1)D_2} + \frac{n D_1}{(m-1)D} \right]^{-1} \]

### B. Bayesian estimation using informative priors based on Weighted Squared Error loss function

The prior distribution of the parameters \((\alpha, \alpha_1, \alpha_2)\) is gamma distribution with hyper-parameters \((a, a_1, a_2, b, b_1, b_2)\) with pdf's as follows (10):

\[ \Pi(a) = \frac{b^a}{\Gamma(a)} a^{a-1} e^{-ba} \quad \Pi(a_1) = \frac{b_1^{a_1}}{\Gamma(a_1)} a_1^{a_1-1} e^{-b_1a_1} \quad \Pi(a_2) = \frac{b_2^{a_2}}{\Gamma(a_2)} a_2^{a_2-1} e^{-b_2a_2} \]

Then the posterior for \((\alpha, \alpha_1, \alpha_2)\) will be as follows:

Since \(x, t\) and \(Z\) are independent random variables The joint posterior distribution For \((\alpha, \alpha_1, \alpha_2)\) can be found as:

\[ P(\alpha, \alpha_1, \alpha_2|x, t, Z) = \frac{Q_{n+a} Q_{m+a_1} Q_{w+a_2}}{\Gamma(n+a) \Gamma(m+a_1) \Gamma(w+a_2)} \alpha^{n+a-1} \alpha_1^{m+a_1-1} \alpha_2^{w+a_2-1} e^{-\alpha} e^{-\alpha_1} e^{-\alpha_2} \]

the posterior distribution for each parameter is:

\[ P(\alpha|x) = \frac{Q_{n+a}}{\Gamma(n+a)} \alpha^{n+a-1} e^{-\alpha} \quad \text{where} \quad Q = -\sum_{i=1}^{n} \ln \left( 1 - e^{-\beta_i} \right) + b \]

\[ P(\alpha_1|t) = \frac{Q_{m+a_1}}{\Gamma(m+a_1)} \alpha_1^{m+a_1-1} e^{-\alpha_1} \quad \text{where} \quad Q_1 = -\sum_{i=1}^{m} \ln \left( 1 - e^{-\beta_1} \right) + b_1 \]

\[ P(\alpha_2|Z) = \frac{Q_{w+a_2}}{\Gamma(w+a_2)} \alpha_2^{w+a_2-1} e^{-\alpha_2} \quad \text{where} \quad Q_2 = -\sum_{k=1}^{w} \ln \left( 1 - e^{-\beta_2} \right) + b_2 \]

the estimated reliability function based on Weighted Square Error Loss Function when the priors are informative is defined as:

\[ \hat{R}_{BN} = \frac{1}{E(R^{-1}|x, t, Z)} \]

Where

\[ E(R^{-1}|x, t, Z) = \int_0^\infty \int_0^\infty \int_0^\infty (1 + 2a^{\alpha}a_1^{\alpha_1}a_2^{\alpha_2} + \alpha^{\alpha_1} \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} + \alpha^{\alpha_1} \alpha_1^{\alpha_1} \alpha_2^{\alpha_2}) da da_1 da_2 \]

\[ E(R^{-1}|x, t, Z) = \frac{Q_{n+a} Q_{m+a_1} Q_{w+a_2}}{\Gamma(n+a) \Gamma(m+a_1) \Gamma(w+a_2)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-\alpha} e^{-\alpha_1} e^{-\alpha_2} da da_1 da_2 \]

By solving the integrations which is kernels of gamma distribution

\[ E(R^{-1}|x, t, Z) = 1 + 2 \frac{(w+1)(w+a_1)}{(m+a_1-1)Q_1} + \frac{(w+a_2)Q_1}{(m+a_1-1)Q_2} + \frac{(w+1)(w+a_1)}{(n-1)(m+a_1-1)Q_2} + \frac{(n)D}{(m-1)Q_2} \]

Substituting equation above in \( \hat{R}_w \) to get the Bayesian estimation using informative prior based on Weighted Square Error Loss Function:
Interval Estimation

The confidence interval can be defined as a numerical range that is expected to contain the true value of an unknown parameter. As for interval estimation; it is the estimate of the unknown parameter within a certain range (period) of values with a certain probability. This probability is called the confidence level and is symbolized by the symbol (1 - estimation error).

To find the estimated confidence interval (interval estimation) of the stress-strength reliability function \( R \), the asymptotic variances of the estimated parameters \( \hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2 \) must be found first; Then the interval estimation of the reliability is generated based on these variances, in this section interval estimation of the stress-strength reliability function of the model \( P(T<X<Z) \) will be found based on estimated reliability by the Maximum Likelihood method and Assuming to be for large samples.

And the formula for the asymptotic variances of reliability function will be found according to the following theorem:

**Theorem**: Let \( T_{1n}, ..., T_{kn} \) be statistics for the parameters \( \theta_1, ..., \theta_k \) such that as \( (n \to \infty) \) then the probability distribution of the difference between the statistics and the parameters is in the following form:

\[
\sqrt{n} [T_{1n} - \theta_1, ..., T_{kn} - \theta_k] \to D N(0, \Sigma)
\]

Where \( \to D \) means "converges in distribution to", and \( \Sigma = (\sigma_{ij})_{k \times k} \) is a matrix with \( k \times k \) dimension, which represent variance - covariance matrix for the estimated parameters and that \( 0 \) represents a zero vector with dimension \( k \times 1 \).

If \( g(T_{1n}, ..., T_{kn}) \) is a function in terms of statistics such that all its first derivatives with respect to parameters \( \theta_1, ..., \theta_k \) exist ; and \( g(\theta_1, ..., \theta_k) \) is a function in terms of the parameters when \( (n \to \infty) \) then the Asymptotic distribution of \( g(T_{1n}, ..., T_{kn}) \) is:

\[
\sqrt{n} [g(T_{1n}, ..., T_{kn}) - g(\theta_1, ..., \theta_k)] \to D N(0, AsyVar(g(T_{1n}, ..., T_{kn})))
\]

Where \( AsyVar(g(T_{1n}, ..., T_{kn})) \) represents the value of the asymptotic variance of function \( g(T_{1n}, ..., T_{kn}) \) which can be found by the formula:

\[
AsyVar(g(T_{1n}, ..., T_{kn})) = [d^T(\theta_{ij}) Asycov(T_{in}, T_{jn}) d(\theta_{ij})]_{i,j=1,2,..,k}
\]

Where \( Asycov(T_{in}, T_{jn}) \) represent variance - covariance matrix for the T\(_{ij}\) statistics. And \( d^T(\theta_{ij}) \) is a row vector with a dimension of \( 1 \times k \) and it represents the derivative of the function in terms of parameters with respect to its parameters:

\[
d^T(\theta_{ij}) = \left[ \frac{\partial g(\theta_1, ..., \theta_k)}{\partial \theta_i} \right]_{i,j=1,2,..,k}
\]

By Applying this theorem to the stress-strength reliability function of the model \( P(T<X<Z) \), the asymptotic variance of the stress-strength reliability function \( (\hat{R}) \) will be:

\[
AsyVar(\hat{R}) = [d^T(\alpha, \alpha_1, \alpha_2) Asycov(\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2) d(\alpha, \alpha_1, \alpha_2)]
\]

Where \( Asycov(\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2) \) represent variance-covariance matrix for \( (\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2) \). And \( d^T(\alpha, \alpha_1, \alpha_2) \) is a row vector with a dimension of \( 1 \times k \) and it represents the derivative of the stress-strength reliability function with respect to \( (\alpha, \alpha_1, \alpha_2) \).

To find the interval estimation of the stress-strength reliability function \( (\hat{R}) \) for the model \( P(T<X<Z) \) based on estimated reliability by the Maximum Likelihood method for large samples and it is necessary to find the variance-covariance matrix for the estimated parameters \( (\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2) \) that have been estimated by Maximum Likelihood method and it can be found using C. R. lower bound note that Maximum likelihood estimators are unbiased \( (E \hat{\alpha} = \alpha, E \hat{\alpha}_1 = \alpha_1, E \hat{\alpha}_2 = \alpha_2) \) for large samples \( (n \to \infty, m \to \infty, W \to \infty) \); As a result \( Asycov(\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2) \) will be (11):

\[
Asycov(\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2) = l^{-1}(\alpha, \alpha_1, \alpha_2)
\]

\( l^{-1}(\alpha, \alpha_1, \alpha_2) \) is the inverse of \( (\alpha, \alpha_1, \alpha_2) \) and can be found as follows:

\[
R_{SW} = \left[ 1 + \frac{(\alpha + a_2)Q_1}{(m + a_1 - 1)Q_2} + \frac{(\alpha + a_2)Q}{(n + a - 1)Q_2} + \frac{(\alpha + a_2 + 1)(\alpha + a_2)Q_1}{(n + a - 1)(m + a_1 - 1)Q_2} + \frac{(\alpha + a)Q_1}{(m + a_1 - 1)Q} \right]^{-1}
\]
The asymptotic variance \( \text{AsyVar} \) of the stress-strength reliability function \( \bar{R} \) of the model \( P(T<X<Z) \) based on estimated reliability by the Maximum Likelihood method will be:

\[
\text{AsyVar}(\bar{R}) = (d^T(\alpha, \alpha_1, \alpha_2) \cdot I^{-1}(\alpha, \alpha_1, \alpha_2) \cdot d(\alpha, \alpha_1, \alpha_2))
\]

Where:

\[
I^{-1}(\alpha, \alpha_1, \alpha_2) = \begin{bmatrix}
\frac{\alpha^2}{n} & 0 & 0 \\
0 & \frac{\alpha_2^2}{m} & 0 \\
0 & 0 & \frac{\alpha_2^2}{w}
\end{bmatrix}
\]

The interval of the stress-strength reliability function for large samples using the reliability function estimated by the maximum likelihood method take the following form:

\[
P[\bar{R}(mle) - Z_{\alpha/2} \cdot \text{AsyVar}(\bar{R}) < R < \bar{R}(mle) + Z_{\alpha/2} \cdot \text{AsyVar}(\bar{R})] = 1 - \text{estimation error}
\]

Then the interval estimation of the stress-strength reliability function will be:

\[
\bar{R}(mle) \pm Z_{\alpha/2} \cdot \text{AsyVar}(\bar{R})
\]

By applying \( \text{AsyVar}(\bar{R}) \) in the equation above; the lower limit and the upper limit will be respectively as follows:

\[
\bar{R}_L = (\bar{R}(mle)) - Z_{\alpha/2} \cdot \begin{bmatrix}
\frac{\alpha^2}{n} \cdot \left(\frac{\partial \bar{R}}{\partial \alpha}\right)^2 + \frac{\alpha_1^2}{m} \cdot \left(\frac{\partial \bar{R}}{\partial \alpha_1}\right)^2 + \frac{\alpha_2^2}{w} \cdot \left(\frac{\partial \bar{R}}{\partial \alpha_2}\right)^2
\end{bmatrix}
\]

\[
\bar{R}_U = (\bar{R}(mle)) + Z_{\alpha/2} \cdot \begin{bmatrix}
\frac{\alpha^2}{n} \cdot \left(\frac{\partial \bar{R}}{\partial \alpha}\right)^2 + \frac{\alpha_1^2}{m} \cdot \left(\frac{\partial \bar{R}}{\partial \alpha_1}\right)^2 + \frac{\alpha_2^2}{w} \cdot \left(\frac{\partial \bar{R}}{\partial \alpha_2}\right)^2
\end{bmatrix}
\]

**Simulation study**

In this section, the simulation study was used to determine best estimator for the stress-strength reliability (S-S.R.) of Exponentiated Inverse Rayleigh distribution from three estimators which are (Maximum likelihood estimator \( \hat{R}(mle) \), Bayesian estimator using Non-informative Jeffrey's prior based on Weighted Square Error Loss Function \( \hat{R}_{R(NW)} \), Bayesian estimation using informative gamma prior based on Weighted Square Error Loss Function \( \hat{R}_{BW} \)), and the mean square error for the estimators had been evaluated with different sample sizes (25,50,100) when \( \lambda = 40, \alpha = 20, \alpha_1 = 18, \alpha_2 = 16, R = 0.1851852 \) and for gamma priors \( a = 1.7, \alpha_1 = 1.5, \alpha_2 = 1.2, b = 0.99, b_1 = 0.81, b_2 = 0.7 \) for
1000 replicates and the simulation study calculated by (R Studio). And to compute the execution of the (S-S.R.) estimator as in steps:
A. Generate random values for $x, t$ and $z$ by the inverse function according to:
$$x = \lambda / \left(-\ln(1 - (1 - u)^{1/2})\right)^{1/2}$$
where $u$ is generated from the uniform distribution.
B. Calculate the mean of the estimators by
$$\bar{\hat{R}} = \frac{\sum_{i=1}^{n} \hat{R}_i}{\text{length}([\hat{R}_i])}$$
C. Evaluate the mean square error (MSE) for the estimators $MSE = \frac{\sum_{i=1}^{n} (\hat{R}_i - \bar{R})^2}{\text{length}([\hat{R}_i])}$.
And the estimator with smallest Mean square error (MSE) considered the best estimator under that size.

Table 1: Simulation results when $\lambda = 40, \alpha = 20, \alpha_1 = 18, \alpha_2 = 16, R = 0.1851852$

<table>
<thead>
<tr>
<th>(n,m,w)</th>
<th>$\hat{R}(mle)$</th>
<th>$\hat{R}_{BW}$</th>
<th>$\hat{R}_{BNW}$</th>
<th>Interval</th>
<th>Lower</th>
<th>Upper</th>
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Discussion

The results of simulation showed that the reliability value is $R = 0.1851852$. The Bayesian estimation using informative priors based on Weighted Square Error Loss Function is the best estimator for the equal sizes, and Bayesian estimation using non-informative priors based on Weighted Square Error Loss Function is the best estimator when the size (w) of the stress sample (Z) larger than the sizes of (X,T), and Maximum Likelihood Estimator is the best estimator for the rest cases.

The experiment was also applied on another values and it showed the same results.

Conclusion

Acknowledgment

The authors are sincerely grateful to the University of Mosul and In this paper, point and interval estimation was presented to estimate the reliability function $P(T<X<Z)$ when each of X, Z and T follows Exponentiated Inverse Rayleigh Distribution with different shape parameters for complete data, the point estimation included maximum likelihood method and Bayesian estimation using informative Gamma priors and non-informative priors based on Weighted Square Error Loss Function (WSELF) for interval estimation confidence interval were estimated for the reliability function $P(T<X<Z)$ based on maximum likelihood estimator of the reliability. Simulation results that appeared confirm that the value of the reliability and confidence intervals is between (0,1) which match the statistical theory and the Bayesian estimation using informative priors based on Weighted Square Error Loss Function is the best estimator for the equal sizes, and Bayesian estimation using non-informative priors based on Weighted Square Error Loss Function is the best estimator when the size (w) of the stress sample (Z) larger than the sizes of (X,T), and Maximum Likelihood Estimator is the best estimator for the rest cases.

College of Computer Sciences and Mathematics for their provided facilities, which helped me very much to improve this work's quality.

Conflict of interest

The authors have no conflict of interest.

References

التقدير النقطي والفتروي لنوىذج القوى-الإجهاد لتوزيع معكوس رايلي الإسـي

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الخلاصة

في هذه الدراسة تم ايجاد صيغة رياضية لدالة موثوقة القوة-الإجهاد لتوزيع معكوس رايلي الإسـي (X), T, Z متغـيـرات شكل مجهولة لمتغير القوة (T) وفاع بين اجهادين (Z) و (X), حيث إن تم تقدير هذه الصيغة باستخدام طرق التقدير الفتراي والتقريبي، في التقدير الفتراي تم تقدير استخدام طريقة الإمكان الأعظم واسـلب بيـز في حال كهن المتغـيـرات الأولية غنية بالدالة. ومع ذلك، تم تقدير فترات الثقة لدالة موثوقة القوة-الإجهاد بالاعتماد على مقدـر دالة الموثوقة بـطريقة الإمكان الأعظم.

تم إجراء دراسة محاكاة بطريقة المونتي كارلو لإيجاد قيم المتغـيـرات وتحديد أفضلية تقديرات التقدير الفتراي بالاعتماد على مجموع متوسط مربعات الخطأ، وقد أظهرت نتائج المحاكاة أفضلية التقدير بإسـلب بيـز في حال كهن المتغـيـرات الأولية غنية بالدالة وتحت دالة خسارة الخطأ الفريذة الموزونة في حال تساوي احجام العينات في حين تكون الإمكانية الإمكانية النتائج في حال كهن حجم المتغـيـرات (Z) أكبر من احجام عينات المتغـيـرات، أما مقدار الإمكان الأعظم فتكون هي الأفضل في بقية الحالات.

الكلمات المفتاحية: تقارير النقطة والفرقة الزمنية الموزونة، نموذج قوى الإجهاد (X), دالة خسارة الخطأ المرني المربع.

توزيع رايلي العكسي الإسـي.