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Bayes Estimator of one parameter Gamma distribution under Quadratic and LINEX Loss Function

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ABSTRACT

In this paper we derive Bayes' estimator for the Scale parameter θ in Gamma distribution when α is known and equal 2, i.e. $X_1, X_2, ..., X_n \sim Ga(2,\theta)$, we take $\alpha = 2$ to estimate one parameter of gamma distribution which is θ (Scale parameter), where gamma distribution is considered as an important model of the life time models . These estimators are obtained depending on squared error and LINEX loss function , Then comparisons of risks for θ under squared and LINEX loss function have been made . Simulation study is given to illustrate that the proposed estimators $\hat{\theta}_{LB}$ is preferable to $\hat{\theta}_{SB}$ for the sample sizes $n = 10,20,30$ from above distribution with parameters $(\alpha = 2, \theta = 1)$ and for all values of "a" $(a = \pm 0.5, \pm 1, \pm 2)$.

مقدر بيز لمعلمة واحدة من توزيع كاما تحت دالة الخسارة التربيعية ودالة خسارة لنكس

المستخلص

تم في هذا البحث اشتقاق مقدرات بيز لمعلمة القياس θ في توزيع كاما عنـدما α معلومة وتساوي (2) هذا يعني $X_1, X_2, ..., X_n \sim Ga(2,\theta)$ ، وقد تم اخـذ $\alpha = 2$ لتقدير معلمة واحدة من توزيع كاما وهي θ (معلمة القياس)، اذ ان توزيـع كامـا يعتبر نموذجا مهما لنماذج الحياة . وقد تم الحصول على هذه المقدرات بالاعتمـاد على دالة الخطأ التربيعية ودالة خسارة لنكس ، ثم تمت بعد ذلك مقارنة الخطورات تحت دالة الخسارة التربيعية ودالة خسارة لنكس . لقد قمنا بدراسة محاكاة وتوضيح ان المقدرات المقترحة $\hat{\theta}_{LB}$ هي مفضلة علـى $\hat{\theta}_{SB}$ ولاحجام عينـات $n = 10,20,30$ من التوزيع اعلاه وبمعلمات $(\alpha = 2, \theta = 1)$ ولجميع قيم "a" $(a = \pm 0.5, \pm 1, \pm 2)$.

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1- Introduction

In Reliability studies the models which are used in life testing include the Exponential, Gamma, Lognormal, …distributions. If the failure is mainly due to aging or wearing out process, then it's reasonable in many applications to choose one of the above mentioned distributions (see Chhikara & Folks (1977), Sinha & Kale (1980), Von Alven (ed.)(1964), Sherif & Smith (1980) .

The Gamma distribution is the most widely used in life experiment , it has probability density function (p.d.f) with two parameters α and θ is :

$$f(x;\alpha,\theta)=\frac{x^{\alpha-1}}{\theta^{\alpha}\Gamma\alpha}\exp(-\frac{x}{\theta}) \qquad x\geq 0 \ , \ \alpha,\theta\geq 0$$

In this paper we will use the gamma distribution with one parameter θ where α is known and equal 2, so it has probability density function (p.d.f) as follows :

$$f(x;\theta)=\frac{x}{\theta^{2}}\exp(-\frac{x}{\theta}) \qquad x\geq 0 \ , \ \alpha,\theta\geq 0 \qquad (1)$$

And the distribution function of X is :

$$F(t)=\int_0^t f(x;\theta)dx=\int_0^t \frac{x}{\theta^{2}}\exp(-\frac{x}{\theta})dx=1-\exp(-\frac{t}{\theta})-\frac{t}{\theta}\exp(-\frac{t}{\theta}) \qquad t\geq 0$$

The Reliability function, the probability of no failure before time t is :

$$R(t)=1-F(t)=[1-\frac{t}{\theta}]*\exp(-\frac{t}{\theta})$$

Then the Hazard function which is the failure rate of a gamma distribution is :

$$H(t)=\frac{f(t)}{R(t)}=\frac{\frac{t}{\theta^{2}}\exp(-\frac{t}{\theta})}{[1-\frac{t}{\theta}]*\exp(-\frac{t}{\theta})}=\frac{t}{\theta(\theta-t)} \qquad (2)$$

In Bayesian estimation, we consider two types of loss functions. The first is squared error loss function (quadratic loss) which classified as a symmetric function and associates equal importance to the losses for overestimation and underestimation of equal magnitude. The second is the LINEX (linear-

exponential where the name LINEX is justified by the fact that is this loss function rises approximately linearly on one side of zero and approximately exponentially on the other side) loss function which is asymmetric, was introduce by Varin (1975). These loss functions were widely used by several authors; among them are Rojo (1987), Basu and Ebrahimi (1991), Pandy(1997), Soliman (2000) and Nassar and Eissa (2004).
The quadratic loss for Bayes estimate of a parameter θ, The posterior mean assuming that exists, denoted by θ_s. The LINEX loss function may be expressed as :

$$L(\Delta) \propto \exp(c\Delta) - c\Delta - 1 \qquad c \neq 0 \tag{3}$$

Where $\Delta = \hat{\theta} - \theta$. The sign and magnitude of the shape parameter c reflects the direction and degree of asymmetry, respectively.
(If $c > 0$), the overestimation is more serious than underestimation, and vice-versa). For c closed to zero, the LINEX loss is approximately squared error loss and therefore almost symmetric.
The posterior expectation of the LINEX loss function equation (3) is :

$$E[L(\hat{\theta} - \theta)] \propto \exp(c\hat{\theta})E[\exp(-c\theta)] - c(\hat{\theta} - E(\theta)) - 1 \tag{4}$$

By result of Zellner(1986), the Bayes estimator of θ, denoted by $\hat{\theta}_L$ under the LINEX loss is the value $\hat{\theta}$ which minimize (4), is given by :

$$\hat{\theta}_L = -\frac{1}{c}\ln\{E[\exp(-c\theta)]\} \tag{5}$$

When the expectation $E[\exp(-c\theta)]$ exists and finite [see Calabria and Pulcini(1996)].

2- LINEX loss function and its properties

Thompson and Basu (1996) identified a family of loss functions $L(\Delta)$, where Δ is either the estimation error $(\hat{\theta} - \theta)$ or the relative estimation error $(\hat{\theta} - \theta)/\theta$, such that

- $L(0) = 0$
- $L(\Delta) > (<) L(-\Delta) > 0$ for all $\Delta > 0$

- $L(\cdot)$ is twice differentiable with $L'(0)=0$ and $L''(\Delta)>0$ for all $\Delta \neq 0$.
- $0 < L'(\Delta) > (<) - L'(-\Delta) > 0$ for all $\Delta > 0$.

Such loss function is useful whenever the actual losses are nonnegative , increases with estimation error, overestimation is more (less) serious than under estimation of the same magnitude and losses increase at a faster (slower) rate with overestimation error .

Considering the loss function

$$L^*(\Delta) \propto b\exp(a\Delta) + c\Delta + d$$

and with the restriction $L^*(0)=0$, $(L^*)'(0)=0$, we get $d=-b$ and $c=-ab$,see Thompson and Basu (1996). The resulting loss function is:

$$L^*(\Delta) \propto b[\exp(a\Delta) - a\Delta - 1] \qquad (6)$$

Which is considered as a function of θ and $\hat{\theta}$, is called the LINEX loss function, a and b are constants with $b>0$ so that the loss function is nonnegative . The shape of the LINEX loss function (6) is determined by the constant a , and the value of b will be taken equal one (i.e. $b=1$) .

In figure 1, values of $\exp(a\Delta) - a\Delta - 1$ are plotted against Δ for selected values of a. It is seen that for $a>0$, the curve rises almost exponentially when $\Delta > 0$ and almost linearly when $\Delta < 0$. on the other hand, if $a<0$ the function rises almost exponentially when $\Delta < 0$ and almost linearly when $\Delta > 0$. So the sign of a reflects the direction of asymmetry, $a>0 (a<0)$ if overestimation is more (less) serious than underestimation; and its magnitude reflects the degree of asymmetry. The important view is that for small values of $|a|$ the function is almost symmetric and not far from a squared error loss (SEL).

The expanding $\exp(a\Delta) \approx 1 + a\Delta + a^2\Delta^2/2$, $L(\Delta) \approx a^2\Delta^2/2$, a SEL function.

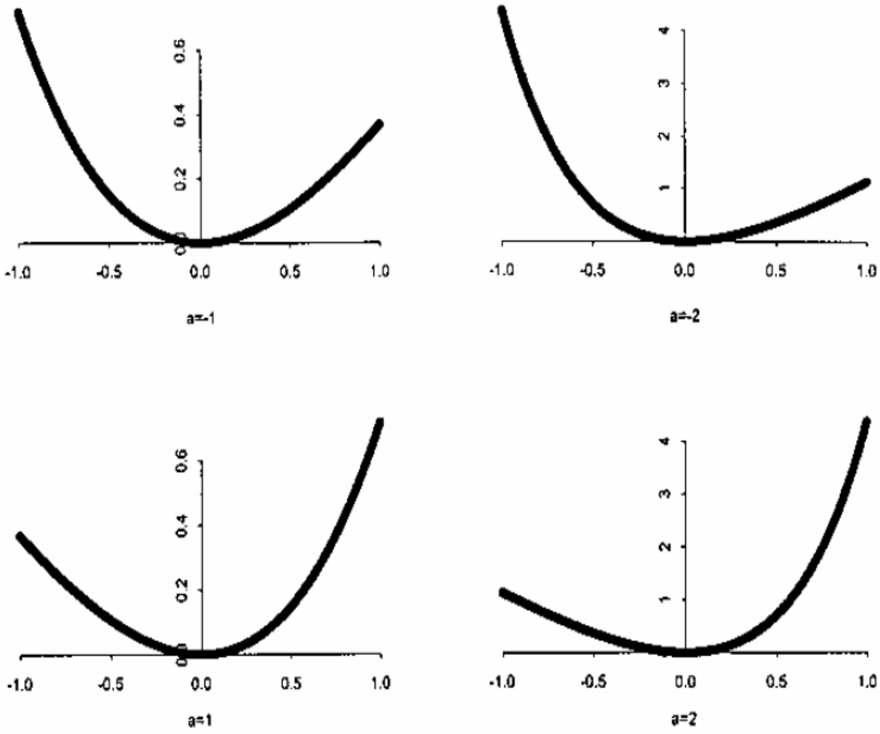

Figure 1. The LINEX loss function.

3- Bayes' Estimator for θ

In this section we derived the estimation of unknown parameter θ of the Gamma distribution based on random sample of size n .The likelihood function is given by :

$$L(X|\theta)=\prod_{i=1}^{n} f(X_i|\theta)=\frac{\prod_{i=1}^{n} X_i}{\theta^{2n}}\exp\left(\frac{-\sum_{i=1}^{n} X_i}{\theta}\right) \tag{7}$$

The natural logarithm of the likelihood function (7) :

$$\ell=\ln L(X|\theta)=\sum \ln X_i-2n\ln\theta-\frac{\sum X_i}{\theta}$$

$$\Rightarrow \hat{\theta}_{ML}=\frac{\sum X_i}{2n} \tag{8}$$

Here we consider the non-informative prior $f(\theta)$ which is derived from (7) as follows :

$$\frac{\partial^2 \ell}{\partial\theta^2}=\frac{2n}{\theta^2}-\frac{2\sum X_i}{\theta^3} \qquad \Rightarrow F.I=-E\left[\frac{-\partial^2\ell}{\partial\theta^2}\right]=\frac{2n}{\theta^2}$$

Then the prior distribution of θ is :

$$f(\theta) = \sqrt{F.I} = \sqrt{2n/\theta^2} \propto 1/\theta \tag{9}$$

By combining the prior distribution $f(\theta)$ in (9) with the likelihood function $L(X|\theta)$, using Bayes theorem we get the posterior distribution :

$$\pi(\theta|X) \propto L(X|\theta).f(\theta)$$

$$\Rightarrow \pi(\theta|X) \propto \theta^{-(2n+1)} \exp(-\frac{\sum X_i}{\theta})$$

Which is the kernel distribution of Inverse gamma distribution, then

$$\pi(\theta|X) = \frac{(\sum X_i)^{2n}}{\Gamma 2n} \theta^{-(2n+1)} \exp(-\frac{\sum X_i}{\theta}) \qquad \theta > 0 \tag{10}$$

And from (10), the expectation of the posterior distribution above, we get :

$$E(\theta|X) = \hat{\theta}_{Bayes} = \frac{\sum X_i}{2n-1} \tag{11}$$

we observe that, if n is large and approach to ∞, then Bayes' estimator will be same as the estimator of Maximum Likelihood, i.e.

$$\text{if } n \to \infty \ , \Rightarrow \hat{\theta}_{Bayes} = \hat{\theta}_{ML} \ .$$

4- Bayes' Estimator of θ under squared error loss function

Under squared error function $L(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2$, and by using (10),we have

$$L(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2$$

Then we minimize

$$\int_{\theta} (\hat{\theta}-\theta)^2 \pi(\theta|X) d\theta$$

The Bayes' estimator of θ denoted by $\hat{\theta}_{SB}$ (SB=Squared Bayes) is the posterior mean

$$\hat{\theta}_{SB} = E_{\pi}(\theta) = \frac{\sum X_i}{2n-1} \tag{12}$$

5- Bayes' Estimator of θ under LINEX loss function

Suppose that $\Delta = \frac{\hat{\theta}}{\theta} - 1$, where $\hat{\theta}$ is an estimate of θ . consider the loss function :

$$L(\Delta) = \exp(a\Delta) - a\Delta - 1 \qquad a \neq 0$$

Under this LINEX loss function ,the posterior mean of the loss function $L(\Delta)$ with respect to $\pi(\theta|X)$ in (10) is :

$$E[L(\Delta)] = \int_0^\infty \{\exp[a(\frac{\hat{\theta}}{\theta} - 1)] - a(\frac{\hat{\theta}}{\theta} - 1) - 1\}\pi(\theta|X)d\theta \qquad (13)$$

By using the integration by part we get :

$$E[L(\Delta)] = \int_0^\infty \{\exp[a(\frac{\hat{\theta}}{\theta} - 1)]\}\pi(\theta|X)d\theta - \int_0^\infty a(\frac{\hat{\theta}}{\theta} - 1)\pi(\theta|X)d\theta - \int_0^\infty \pi(\theta|X)d\theta$$

$$= \exp(-a)E[\exp\{a(\frac{\hat{\theta}}{\theta})\}] - aE[(\frac{\hat{\theta}}{\theta}) - 1] - 1 \qquad (14)$$

The value of $\hat{\theta}$ that minimizes the posterior expectation of the loss function $L(\Delta)$ denoted by $\hat{\theta}_{LB}$ (LB=<u>L</u>INEX <u>B</u>ayes) is obtained by solving the equation :

$$\frac{\partial E[L(\Delta)]}{\partial \hat{\theta}} = E[e^{-a}\frac{a}{\theta}\exp(a(\frac{\hat{\theta}}{\theta})] - aE(\frac{1}{\theta}) = 0$$

$$\Rightarrow E[\frac{1}{\theta}\exp\{a(\frac{\hat{\theta}}{\theta})\}] = e^{a}E(\frac{1}{\theta}) \qquad (15)$$

Provided that all expectations exist and finite, then we will use (10) and (15) to find the expectations and get the optimal estimate for θ,

$$E[\frac{1}{\theta}\exp\{a(\frac{\hat{\theta}}{\theta})\}] = \int_0^\infty \frac{1}{\theta}\exp\{a(\frac{\hat{\theta}}{\theta})\}\pi(\theta|X)d\theta$$

$$= \frac{(\sum X_i)^{2n}}{\Gamma 2n}\int_0^\infty \theta^{-(2n+2)}\exp\{a(\frac{\hat{\theta}}{\theta})\}\exp(\frac{-\sum X_i}{\theta})d\theta$$

$$= \frac{(\sum X_i)^{2n}}{\Gamma 2n}\int_0^\infty \theta^{-(2n+2)}\exp\{-\frac{1}{\theta}(\sum X_i - a\hat{\theta})d\theta$$

$$\Rightarrow E[\frac{1}{\theta}\exp\{a(\frac{\hat{\theta}}{\theta})\}] = \frac{(\sum X_i)^{2n}}{\Gamma 2n} * \frac{\Gamma 2n+1}{(\sum X_i - a\hat{\theta})^{2n+1}} = \frac{2n(\sum X_i)^{2n}}{(\sum X_i - a\hat{\theta})^{2n+1}}$$

And also

$$E(\frac{1}{\theta}) = \int_0^\infty \frac{1}{\theta}\pi(\theta|X)d\theta = \frac{(\sum X_i)^{2n}}{\Gamma 2n}\int_0^\infty \theta^{-(2n+2)}\exp(\frac{-\sum X_i}{\theta})d\theta$$

$$= \frac{(\sum X_i)^{2n}}{\Gamma 2n} * \frac{\Gamma 2n+1}{(\sum X_i)^{2n+1}}$$

$$= \frac{2n}{\sum X_i}$$

Then from (15), we have :

$$\frac{2n(\sum X_i)^{2n}}{(\sum X_i - a\hat{\theta})^{2n+1}} = \frac{2n}{\sum X_i}e^a \quad \Rightarrow \frac{(\sum X_i)^{2n+1}}{(\sum X_i - a\hat{\theta})^{2n+1}} = e^a$$

$$\therefore (1 - \frac{a\hat{\theta}}{\sum X_i})^{2n+1} = e^{-a} \quad \Rightarrow (1 - \frac{a\hat{\theta}}{\sum X_i}) = e^{-\frac{a}{2n+1}}$$

$$\Rightarrow \hat{\theta}_{LB} = \frac{\sum X_i}{a}(1 - e^{-\frac{a}{2n+1}}) \qquad (16)$$

For more information (see Canfield (1970),Varin (1975) and Zellner (1986)) .

6- The Decision Theory, Risk function & Risk Efficiency

The decision theoretic approach begins with a careful definition of all the elements of a decision problem. It is imagined that there is a decision-maker who is to choose an action from a set A. He is to do this based upon observation of a random variable, or data X. This X (typically a vector $X_1,.......,X_n$) has a probability distribution which depends on an unknown parameter θ. Here θ denotes a state of nature. The set of all possible values of θ is the parameter space Θ.

The decision is to be made by a statistical decision function (or rule) d; this is a function which specifies $d(x)$ as the action to be taken when the observed data is $X = x$. On taking action $\hat{\theta} = d(X)$ the decision-maker incurs a loss of $L(\theta,\hat{\theta})$. A good decision function is one that has a small value of the risk function :

$$R(\theta,\hat{\theta}) = E[L(\theta,\hat{\theta})] = \begin{cases} \int_{R_X} L(\hat{\theta},\theta) f(X;\theta) dX \\ \text{or} \\ \sum_{x \in R_X} L(\hat{\theta},\theta) p(X;\theta) \end{cases} \quad (17)$$

Clearly if $R(\theta,\hat{\theta}_1) \le R(\theta,\hat{\theta}_2)$ for all θ and $R(\theta,\hat{\theta}_1) < R(\theta,\hat{\theta}_2)$ for some θ then the risk efficiency for $\hat{\theta}_1$ is better than the risk efficiency for $\hat{\theta}_2$. We say that $\hat{\theta}_2$ is inadmissible. For more information see Weber (2007) .

7- The Risk Efficiency of $\hat{\theta}_{LB}$ with respect to $\hat{\theta}_{SB}$ under squared error loss function

The risk functions of the estimators under squared error loss are denoted by $R(\hat{\theta}_{LB},\theta)$ and $R(\hat{\theta}_{SB},\theta)$,are given by :

$$R_S(\hat{\theta}_{LB},\theta) = \int_0^\infty (\hat{\theta}_{LB} - \theta)^2 f(x_1, x_2, \ldots, x_n | \theta) dx_1 dx_2 \ldots dx_n \quad (18)$$

Let $S = \sum X_i$, because $x_i, i = 1,2\ldots,n$ are identically distributed and independent from gamma distribution with parameters $(2,\theta)$ then $S = \sum X_i \sim Gam(2n,\theta)$, so that :

$$R_S(\hat{\theta}_{LB},\theta) = \int_0^\infty (\hat{\theta}_{LB}^2 - 2\hat{\theta}_{LB}\theta + \theta^2) * \frac{S^{2n-1}}{\theta^{2n}\Gamma 2n} \exp(-\frac{S}{\theta}) dS$$

$$= \int_0^\infty \hat{\theta}_{LB}^2 * \frac{S^{2n-1}}{\theta^{2n}\Gamma 2n} \exp(-\frac{S}{\theta}) dS - 2\int_0^\infty \hat{\theta}_{LB}\theta * \frac{S^{2n-1}}{\theta^{2n}\Gamma 2n} \exp(-\frac{S}{\theta}) dS + \int_0^\infty \theta^2 * \frac{S^{2n-1}}{\theta^{2n}\Gamma 2n} \exp(-\frac{S}{\theta}) dS$$

Thus

$$R_S(\hat{\theta}_{LB},\theta) = \frac{\theta^2[2n(2n+1)(1-e^{-\frac{a}{2n+1}})^2]}{a^2} - \frac{\theta^2[4n(1-e^{-\frac{a}{2n+1}})]}{a} + \theta^2$$

$$\Rightarrow R_S(\hat{\theta}_{LB},\theta) = \theta^2[\frac{2n(2n+1)(1-e^{-\frac{a}{2n+1}})^2}{a^2} - \frac{[4n(1-e^{-\frac{a}{2n+1}})]}{a} + 1] \quad (19)$$

By the same way we can find $R_S(\hat{\theta}_{SB},\theta)$ under squared error loss :

$$R_S(\hat{\theta}_{SB},\theta) = \int_0^\infty (\hat{\theta}_{SB} - \theta)^2 f(x_1, x_2, \ldots, x_n | \theta) dx_1 dx_2 \ldots dx_n \quad (20)$$

Let $S = \sum X_i$ then :

$$R_S(\hat{\theta}_{SB},\theta)=\int_0^\infty(\hat{\theta}_{SB}^2-2\hat{\theta}_{SB}\theta+\theta^2)*\frac{S^{2n-1}}{\theta^{2n}\Gamma 2n}\exp(-\frac{S}{\theta})dS$$

$$=\int_0^\infty\hat{\theta}_{SB}^2*\frac{S^{2n-1}}{\theta^{2n}\Gamma 2n}\exp(-\frac{S}{\theta})dS-2\int_0^\infty\hat{\theta}_{SB}\theta*\frac{S^{2n-1}}{\theta^{2n}\Gamma 2}\exp(-\frac{S}{\theta})dS+\int_0^\infty\theta^2*\frac{S}{\theta^{2n}}\exp(-\frac{S}{\theta})dS$$

Thus

$$R_S(\hat{\theta}_{SB},\theta)=\frac{2n(2n+1)\theta^2}{(2n-1)^2}-\frac{4n\theta^2}{(2n-1)}+\theta^2$$

$$\Rightarrow R_S(\hat{\theta}_{SB},\theta)=\theta^2[\frac{2n(2n+1)}{(2n-1)^2}-\frac{4n}{(2n-1)}+1] \qquad (21)$$

The risk efficiency of $\hat{\theta}_{LB}$ with respect to $\hat{\theta}_{SB}$ under squared error loss function is denoted by :

$$RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})=\frac{R_S(\hat{\theta}_{LB},\theta)}{R_S(\hat{\theta}_{SB},\theta)}=\frac{[\frac{2n(2n+1)(1-e^{-\frac{a}{2n+1}})^2}{a^2}-\frac{[4n(1-e^{-\frac{a}{2n+1}})]}{a}+1]}{[\frac{2n(2n+1)}{(2n-1)^2}-\frac{4n}{(2n-1)}+1]} \qquad (22)$$

8- The Risk Efficiency of $\hat{\theta}_{LB}$ with respect to $\hat{\theta}_{SB}$ under LINEX loss function

The risk functions of the estimators $\hat{\theta}_{LB}$ and $\hat{\theta}_{SB}$ under LINEX loss are denoted by $R_L(\hat{\theta}_{LB},\theta)$ and $R_L(\hat{\theta}_{SB},\theta)$, where the subscript L denotes risk relative to LINEX loss and are given as follows :

$$R_L(\hat{\theta}_{LB},\theta)=\int_0^\infty\{e^{a[(\frac{\hat{\theta}_{LB}}{\theta})-1]}-a[(\frac{\hat{\theta}_{LB}}{\theta})-1]-1\}f(x_1,x_2,\ldots\ldots,x_n|\theta)dx_1dx_2\ldots\ldots dx_n \qquad (23)$$

Let $S=\sum X_i$, and as a mentioned above $S=\sum X_i\sim Gam(2n,\theta)$, so that :

$$R_L(\hat{\theta}_{LB},\theta)=\int_0^\infty\{e^{a[(\frac{\hat{\theta}_{LB}}{\theta})-1]}-a[(\frac{\hat{\theta}_{LB}}{\theta})-1]-1\}*\frac{S^{2n-1}}{\theta^{2n}\Gamma 2n}\exp(-\frac{S}{\theta})dS$$

$$=\int_0^\infty e^{a[(\frac{\hat{\theta}_{LB}}{\theta})-1]}*\frac{S^{2n-1}}{\theta^{2n}\Gamma 2n}\exp(-\frac{S}{\theta})dS-\int_0^\infty a(\frac{\hat{\theta}_{LB}}{\theta})*\frac{S^{2n-1}}{\theta^{2n}\Gamma 2n}\exp(-\frac{S}{\theta})dS+\int_0^\infty a*\frac{S^{2n-1}}{\theta^{2n}\Gamma 2n}\exp(-\frac{S}{\theta})dS-\int_0^\infty\frac{S^{2n-1}}{\theta^{2n}\Gamma 2n}\exp(-\frac{S}{\theta})dS$$

Thus

$$R_L(\hat{\theta}_{LB},\theta)=e^{-\frac{a}{2n+1}}-2n(1-e^{-\frac{a}{2n+1}})+a-1 \qquad (24)$$

By the same way we can find $R_L(\hat{\theta}_{SB},\theta)$ under squared error loss

$$R_L(\hat{\theta}_{sB},\theta)=\int_0^{\infty}\{e^{a[(\frac{\hat{\theta}_{SB}}{\theta})-1]}-a[(\frac{\hat{\theta}_{SB}}{\theta})-1]-1\}f(x_1,x_2,.....,x_n|\theta)dx_1dx_2......dx_n \qquad (25)$$

Let $S=\sum X_i$ then :

$$R_L(\hat{\theta}_{SB},\theta)=\int_0^{\infty}\{e^{a[(\frac{\hat{\theta}_{SB}}{\theta})-1]}-a[(\frac{\hat{\theta}_{SB}}{\theta})-1]-1\}*\frac{S^{2n-1}}{\theta^{2n}\Gamma 2n}\exp(-\frac{S}{\theta})dS$$

$$=\int_0^{\infty}e^{a[(\frac{\hat{\theta}_{SB}}{\theta})-1]}*\frac{S^{2n-1}}{\theta^{2n}\Gamma 2n}\exp(-\frac{S}{\theta})dS-\int_0^{\infty}a(\frac{\hat{\theta}_{SB}}{\theta})*\frac{S^{2n-1}}{\theta^{2n}\Gamma 2n}\exp(-\frac{S}{\theta})dS+\int_0^{\infty}a*\frac{S^{2n-1}}{\theta^{2n}\Gamma 2n}\exp(-\frac{S}{\theta})dS-\int_0^{\infty}\frac{S^{2n-1}}{\theta^{2n}\Gamma 2n}\exp(-\frac{S}{\theta})dS$$

Thus

$$R_L(\hat{\theta}_{SB},\theta)=e^{-a}(1-\frac{a}{2n-1})^{-2n}-\frac{a}{2n-1}-1 \qquad (26)$$

The risk efficiency of $\hat{\theta}_{LB}$ with respect to $\hat{\theta}_{SB}$ under LINEX loss function is denoted by :

$$RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})=\frac{R_L(\hat{\theta}_{LB},\theta)}{R_L(\hat{\theta}_{SB},\theta)}=\frac{e^{-\frac{a}{2n+1}}-2n(1-e^{-\frac{a}{2n+1}})+a-1}{e^{-a}(1-\frac{a}{2n-1})^{-2n}-\frac{a}{2n-1}-1} \qquad (27)$$

9- Numerical Example

We generated $N=500$ samples of sizes $n=10,20,30$ from equation (1) with $\theta=1$, we used Minitab to generate these samples and we take randomly the samples of size $n=10,20,30$, respectively and then the risk functions are computed for the estimators $\hat{\theta}_{LB}$ and $\hat{\theta}_{SB}$ under the LINEX loss function and squared error loss , and also computed the risk efficiency to compare between the LINEX loss and squared error loss function to check which estimator is inadmissible under theses functions . The results are explained in tables from 1 to 6 .

Table 1 : the estimators $\hat{\theta}_{LB}$, $\hat{\theta}_{SB}$, the risk efficiencies $RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$, $RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$ under the prior $f(\theta)$ for the value of $a = 2$

n	$\hat{\theta}_{SB}$	$\hat{\theta}_{LB}$	$R_L(\hat{\theta}_{LB},\theta)$	$R_L(\hat{\theta}_{SB},\theta)$
10	1.0511	0.9071	0.0911	0.1465
20	1.0361	0.9619	0.0443	0.0600
30	1.02	0.9760	0.0324	0.0375
n	$R_S(\hat{\theta}_{LB},\theta)$	$R_S(\hat{\theta}_{SB},\theta)$	$RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$	$RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$
10	0.0495	0.0581	1.6081	1.1737
20	0.025	0.027	1.3544	1.08
30	0.0166	0.0175	1.1574	1.0542

Table 2 : the estimators $\hat{\theta}_{LB}$, $\hat{\theta}_{SB}$, the risk efficiencies $RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$, $RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$ under the prior $f(\theta)$ for the value of $a = 1$

n	$\hat{\theta}_{SB}$	$\hat{\theta}_{LB}$	$R_L(\hat{\theta}_{LB},\theta)$	$R_L(\hat{\theta}_{SB},\theta)$
10	1.2042	0.5320	0.0234	0.0318
20	1.1819	0.5553	0.0121	0.0139
30	1.1766	0.5644	0.0082	0.0088
n	$R_S(\hat{\theta}_{LB},\theta)$	$R_S(\hat{\theta}_{SB},\theta)$	$RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$	$RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$
10	0.0481	0.0581	1.3589	1.2079
20	0.0245	0.027	1.1487	1.1020
30	0.0165	0.0175	1.0731	1.0606

Table 3 : the estimators $\hat{\theta}_{LB}$, $\hat{\theta}_{SB}$, the risk efficiencies $RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$, $RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$ under the prior $f(\theta)$ for the value of $a = 0.5$

n	$\hat{\theta}_{SB}$	$\hat{\theta}_{LB}$	$R_L(\hat{\theta}_{LB},\theta)$	$R_L(\hat{\theta}_{SB},\theta)$
10	1.1072	0.2474	0.0059	0.0075
20	1.0859	0.2566	0.00305	0.00344
30	1.0738	0.2585	0.0020	0.0022
n	$R_S(\hat{\theta}_{LB},\theta)$	$R_S(\hat{\theta}_{SB},\theta)$	$RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$	$RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$
10	0.0478	0.0581	1.2711	1.2154
20	0.0244	0.027	1.1278	1.1065
30	0.0164	0.0175	1.1000	1.0670

Table 4 : the estimators $\hat{\theta}_{LB}$, $\hat{\theta}_{SB}$, the risk efficiencies $RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$, $RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$ under the prior $f(\theta)$ for the value of $a = -2$

n	$\hat{\theta}_{SB}$	$\hat{\theta}_{LB}$	$R_L(\hat{\theta}_{LB},\theta)$	$R_L(\hat{\theta}_{SB},\theta)$
10	1.0409	0.9881	0.0984	0.1035
20	1.0376	1.0115	0.0495	0.0502
30	1.0292	1.0119	0.0332	0.0336
n	$R_S(\hat{\theta}_{LB},\theta)$	$R_S(\hat{\theta}_{SB},\theta)$	$RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$	$RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$
10	0.0499	0.0581	1.0518	1.1643
20	0.0240	0.027	1.0141	1.1250
30	0.0166	0.0175	1.0120	1.0542

Table 5 : the estimators $\hat{\theta}_{LB}$, $\hat{\theta}_{SB}$, the risk efficiencies $RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$, $RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$ under the prior $f(\theta)$ for the value of $a=-1$

n	$\hat{\theta}_{SB}$	$\hat{\theta}_{LB}$	$R_L(\hat{\theta}_{LB},\theta)$	$R_L(\hat{\theta}_{SB},\theta)$
10	1.2081	1.1194	0.0242	0.0269
20	1.1796	1.1359	0.0123	0.0130
30	1.1121	1.0845	0.0083	0.0085
n	$R_S(\hat{\theta}_{LB},\theta)$	$R_S(\hat{\theta}_{SB},\theta)$	$RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$	$RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$
10	0.0482	0.0581	1.1115	1.2053
20	0.0245	0.027	1.0569	1.1020
30	0.0164	0.0175	1.0240	1.0670

Table 6 : the estimators $\hat{\theta}_{LB}$, $\hat{\theta}_{SB}$, the risk efficiencies $RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$, $RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$ under the prior $f(\theta)$ for the value of $a=-0.5$

n	$\hat{\theta}_{SB}$	$\hat{\theta}_{LB}$	$R_L(\hat{\theta}_{LB},\theta)$	$R_L(\hat{\theta}_{SB},\theta)$
10	1.1679	1.0693	0.0060	0.0070
20	1.1347	1.0859	0.0030	0.0032
30	1.1241	1.0917	0.00205	0.00216
n	$R_S(\hat{\theta}_{LB},\theta)$	$R_S(\hat{\theta}_{SB},\theta)$	$RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$	$RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$
10	0.0477	0.0581	1.1666	1.2180
20	0.0244	0.027	1.0666	1.1065
30	0.0165	0.0175	1.0536	1.0606

10- Conclusion

1- From tables (1-6), we observe that the risk efficiency $RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$ is greater than 1, which means that the proposed estimators $\hat{\theta}_{LB}$ is preferable to $\hat{\theta}_{SB}$ for the sample sizes $n=10,20,30$ from gamma distribution with parameters $(\alpha=2,\theta=1)$ and for all values of "a" $(a=\pm0.5,\pm1,\pm2)$.

2- A symmetric loss function is more appropriate than Squared error loss function .

3- We note that the risk efficiency $RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$ is greater than the risk efficiency $RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$ for all positive values of a , and the risk efficiency $RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$ is greater than the risk efficiency $RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$ for all negative values of a and for all sample sizes $n = 10,20,30$.

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