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On the Discrimination between the Inverse Gaussian and Lognormal Distributions

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ABSTRACT

Both inverse Gaussian and lognormal distributions have been used among many well-known failure time distributions with positively skewed data. The problem of selecting between them is considered. The logarithm of maximum likelihood ratio has been used as a test for discriminating between these two distributions. The test has been carried out on nine different real data sets and three simulated data sets.

Keywords: Inverse Gaussian distribution, lognormal distribution, Ratio maximum likelihood, Discrimination.

التمييز بين توزيع كاوس المعكوس والتوزيع اللوغارتمي الطبيعي

الملخص

يعتبر توزيع كاوس المعكوس والتوزيع اللوغارتمي الطبيعي من توزيعات اوقات الفشل في حالة كون البيانات ذات التواء موجب. تهدف الدراسة الى اجراء تمييز بين التوزيعين عندما تخضع البيانات لكلا التوزيعين عن طريق استخدام اختبار نسبة الامكان الاعظم حيث تم اجراء الدراسة على تسعة مجاميع مختلفة من البيانات الحقيقية التي تتبع كلا التوزيعين وكذلك ثلاثة مجاميع من البيانات المولدة بطريقة المحاكاة.

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1-Introducation

It is well known that the inverse Gaussian distribution (IG) and lognormal distribution (LOGN) are used to analyze asymmetric positive data. In reliability and survival analysis we need these distributions on modeling the failure time data. Sometimes we see that both distributions fit our data. So, the question is: Which one will be more preferable than the other?. To answer this question, we use in this paper the likelihood ratio test to discriminate between the IG and LOGN distributions. Nine data sets have been taken to prove our test. Discriminating between any two general probability distribution function was studied by Atkinson (1969, 1970), Dumonceaux et al (1973), Dumonceaux and Antle (1973), and Kundu and Manglick (2004, 2005).

This paper is organized as follows. Section 2 and section 3 show the properties of the IG and LOGN distributions, respectively. In section 4 the description of the likelihood ratio test is mentioned. Nine data sets are analyzed in section 5.

2-The Inverse Gaussian Distribution

The inverse Gaussian distribution is used to model non-negative skewed data. This distribution referred to the theory of Brownian motion because the distribution of the first passage time of a Brownian motion belongs to the inverse Gaussian (Cklikara and Floks 1988).

Inverse Gaussian distribution has many applications and uses especially in reliability (survival analysis), and in the area on natural and social sciences. Since it is a positively skewed distribution, it has advantage over some other skewed distributions like lognormal, gamma, and Weibull.

The p.d.f of an inverse Gaussian r.v X is

$$f(X;\mu,\lambda)=\left(\frac{\lambda}{2\pi}\right)^{1/2} x^{-3/2}\,\mathrm{Exp}\left[-\frac{\lambda}{2\mu^2}\frac{(X-\mu)^2}{X}\right] \quad , \quad X>0 \qquad ..\ (2.1)$$

where $\mu>0$ and $\lambda>0$. The parameter μ represents the mean of the distribution and λ represents the scale parameter. There are three other forms of (2.1) (Tweedie 1957).
The likelihood function of (2.1) is

$$L(\mu,\lambda)=\left(\frac{\lambda}{2\pi}\right)^{n/2}\prod_{i=1}^{n} x_i^{-3/2}\,\mathrm{Exp}\left[-\frac{\lambda}{2\mu^2}\sum_{i=1}^{n}\left\{\frac{(X_i-\mu)^2}{X_i}\right\}\right] \quad \ldots\ldots\ldots\ (2.2)$$

And the natural logarithm of (2.2) is,

$$\mathrm{Ln}\,L(\mu,\lambda)=\frac{n}{2}\ln\lambda-\frac{n}{2}\ln(2\pi)-\frac{3}{2}\ln\left(\prod_{i=1}^{n}x_i\right)-\frac{\lambda}{2\mu^2}\sum_{i=1}^{n}x_i+\frac{n\lambda}{\mu}-\frac{\lambda}{2}\sum_{i=1}^{n}\left(\frac{1}{x_i}\right) \quad (2.3)$$

From (2.3) one can obtain the m.l.e for μ and λ (Tweedie 1956) as in the following:

$$\hat{\mu}=\bar{x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2.4)$$

$$\hat{\lambda}=\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{x_i}-\frac{1}{\bar{x}}\right)\right]^{-1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2.5)$$

3- The Lognormal Distribution

The lognormal distribution is considered as one of the most popular distributions for modeling nonnegative skewed data. The p.d.f of a lognormal r.v X is

$$f(x;\theta,\sigma^2)=\frac{1}{x\sqrt{2\pi\sigma^2}}\,\mathrm{EXP}\left[-\frac{1}{2\sigma^2}(\ln x-\theta)^2\right] \quad , \quad x>0 \qquad ..\ (3.1)$$

where θ is the scale parameter, $-\infty<\theta<\infty$, and σ^2 is the shape parameter, $\sigma^2>0$.

The likelihood function of the lognormal p.d.f is,

$$L(\theta,\sigma^2)=\prod_{i=1}^{n}(\frac{1}{x_i})\,(2\pi)^{-n/2}\,(\sigma^2)^{-n/2}\,EXP\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln x_i-\theta)^2\right] \quad (3.2)$$

The natural logarithm of (3.2) is,

$$Ln\,L(\theta,\sigma^2)=\sum_{i=1}^{n}\ln x_i-\frac{n}{2}\ln(2\pi)-\frac{n}{2}\ln(\sigma^2)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln x_i-\theta)^2 \quad (3.3)$$

By solving $\frac{\partial \ln(\theta,\sigma^2)}{\partial\theta}=0$ and $\frac{\partial \ln(\theta,\sigma^2)}{\partial\sigma^2}=0$ for (3.3), one can get the m.l.e for θ and σ^2 as:

$$\hat{\theta}=\frac{1}{n}\sum_{i=1}^{n}\ln x_i \quad \text{……. (3.4)}$$

$$\sigma^2=\frac{1}{n}\sum_{i=1}^{n}(\ln x_i-\theta)^2 \quad \text{………(3.5)}$$

4- Likelihood Ratio Test

A likelihood ratio test (LRT) is a statistical test relying on statistics computed by taking the ratio of the maximum value of the likelihood function.

Let $X_1, X_2, \ldots\ldots, X_n$ be i.i.d random variables from a known distribution

(with p.d.f). Recall that the likelihood functions and their logarithm are given, then the LRT (let us denote it here by L) is defined as:

$$L=\frac{L_1(\hat{\theta}_1,\hat{\theta}_2)}{L_2(\hat{\lambda}_1,\hat{\lambda}_2)} \quad \text{…. …… … . (4.1)}$$

where $L_1(\hat{\theta}_1,\hat{\theta}_2)$ and $L_2(\hat{\lambda}_1,\hat{\lambda}_2)$ are the likelihood function of a known different p.d.f, and $\hat{\theta}_1,\hat{\theta}_2$, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the m.l.e of θ_1,θ_2, λ_1 and $\hat{\lambda}_2$, respectively.

Now, from our problem, we rewrite (4.1) as:

$$L = \frac{L_{IG}(\hat{\mu}, \hat{\lambda})}{L_{LOGN}(\hat{\theta}, \hat{\sigma}^2)} \qquad \text{………... (4.2)}$$

By taking the natural logarithm of (4.2) and from (2.3), (2.4), (2.5), (3.3), (3.4), and (3.5), one gets

$$\ln L = \frac{n}{2}\left\{\ln(\hat{\lambda}) - 3\ln\overline{G} + \hat{\lambda}\left(\frac{1}{\overline{X}} - \frac{1}{\overline{H}}\right) - \frac{2}{n}\sum_{i=1}^{n}\ln\left(\frac{1}{X_i}\right) + \ln(\hat{\sigma}^2) + 1\right\}$$

………..(4.3)

where $\overline{X}$, $\overline{G}$ and $\overline{H}$ are the arithmetic, geometric, and harmonic mean, respectively.

The hypothesis test will be:

H_0 = The data belong to IG distribution.
H_1 = The data belong to LOGN distribution.

Our decision to choose whether the data belong to the IG or to the LOGN distribution is based on the value of (4.3). If $\ln L > 0$ we choose IG distribution as fitted to the data, elsewhere ($\ln L < 0$) we prefer LOGN distribution as fitted to the data.

5- Analysis of Data

In this section we have taken nine data sets and three simulated data in order to apply the formula (3.4) to discriminate between the two mentioned distributions.

5-1 Real Data Analysis

5.1.1 Data set (1)

This data set refers to (Von Alven, 1964). It represents the active repair times (in hours) for an airborne communication transceiver, the data are:
.2,.3,.5,.5,.5,.5,.6,.6,.7,.7,.7,.8,.8,1,1,1,1,1.1,1.3,1.5,1.5,1.5,1.5,2,2,2.2,2.5,2.7,3,3,3.3,3.3,4,4,4.5,4.7,5,5.4,5.4,7,7.5,8.8,9,10.3,22,24.5.

Table (1): The m.l.e for both distribution parameters and Kolmogrove- Smirnove (K-S) statistic

IG	LOGN
$\hat{\mu} = \mathbf{3.6065}$	$\hat{\theta} = \mathbf{0.6588}$
$\hat{\lambda} = \mathbf{1.658}$	$\hat{\sigma} = \mathbf{1.1018}$
K-S = 0.0578	K-S = 0.0866

Both K-S values are significant (i.e. the data belong to both distributions). But the value of $\ln L$ is 0.957 > 0, therefore the IG distribution is more suitable than LOGN distribution. Also, the K-S distance of IG is less than the K-S of LOGN.

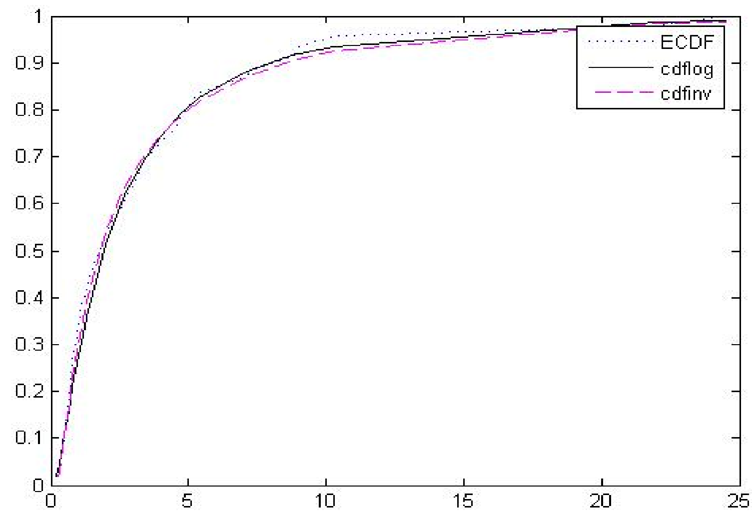

Figure 1: The CDF for both distributions and the ECDF (Kolmogrove- Smirnove CDF) for data set (1).

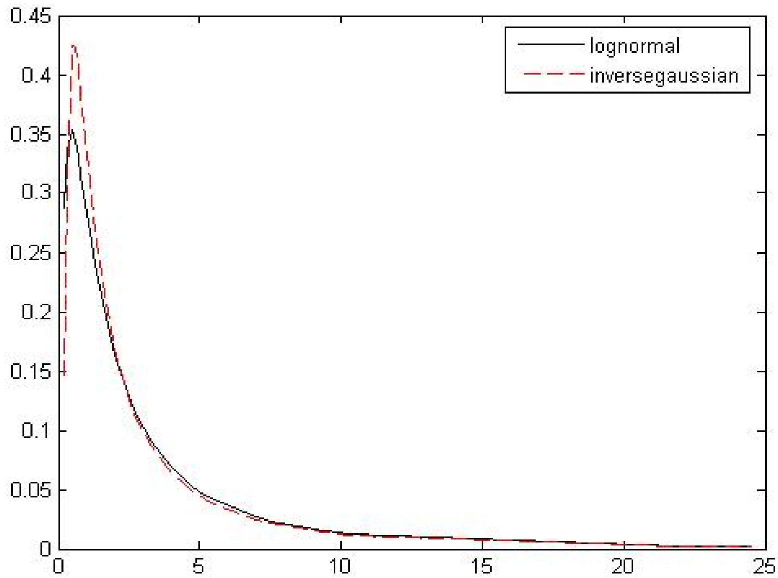

Figure 2: The p.d.f for both distributions for data set (1).

5.1.2 Data set (2)

This data represent the test on endurance of deep groove ball bearings (Lawless, 2003)
17.88,28.92,33,41.52,42.12,45.60,48.48,51.84,51.96,54.12,55.56,67.80,68.64,68.64,68.88,84.12,93.12,98.64,105.12,105.84,127.92,128.04,173.40

Table (2): The m.l.e for both distribution parameters and Kolmogrove- Smirnove (K-S) statistic

IG	LOGN
$\hat{\mu} = \mathbf{72.2243}$	$\hat{\theta} = \mathbf{4.1505}$
$\hat{\lambda} = \mathbf{231.6741}$	$\hat{\sigma} = \mathbf{0.52168}$
K-S = 0.088	K-S = 0.089

The value of $\ln L$ is – 0.0764 < 0, that is, the LOGN is best fitted to these data than IG. Despite that the K-S test values assumed that these data belong to both distributions.

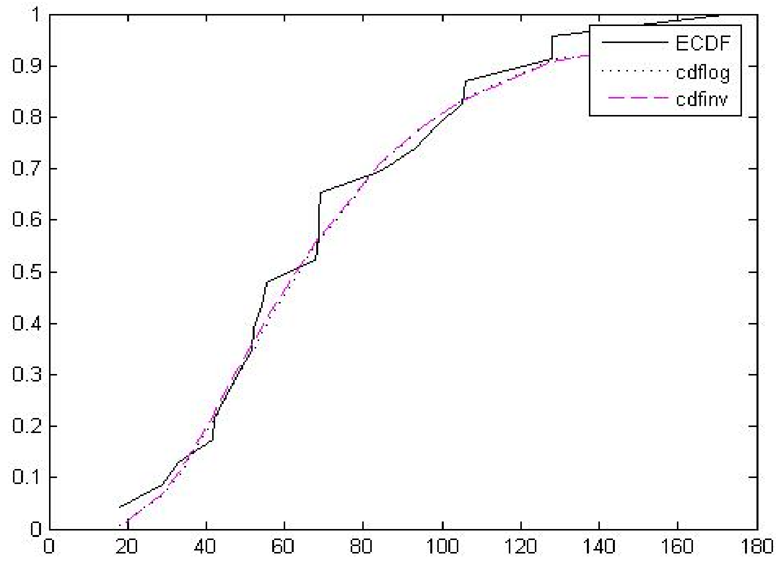

Figure 3: The CDF for both distributions and the ECDF (Kolmogrove- Smirnove CDF) for data set (2)

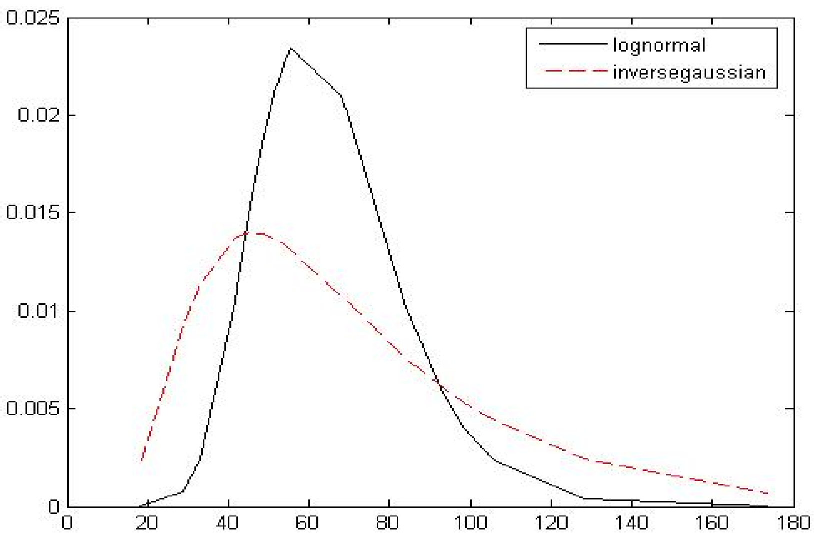

Figure 4: The p.d.f for both distributions for data set (2)

5.1.3 Data set (3)

The third data set (Linhart and Zucchini, 1956) represents the failure times of the air conditioning system of an airplane. The data are:
1,3,5,7,11,11,11,12,14,14,14,16,16,20,21,23,42,47,52,62,71,71,87,90,95,120,120,225,246,261

Table (3): The m.l.e for both distribution parameters and Kolmogrove- Smirnove (K-S) statistic

IG	LOGN
$\hat{\mu} = 59.6$	$\hat{\theta} = 3.3581$
$\hat{\lambda} = 13.7613$	$\hat{\sigma} = 1.3192$
K-S = 0.1944	K-S = 0.127

Here both K-S tests assumed that these data are distributed IG and LOGN. The $\ln L$ value is -2.7336 < 0, so we choose LOGN distribution as the preferred distribution. Based on the K-S values also we prefer to choose the LOGN distribution over IG distribution.

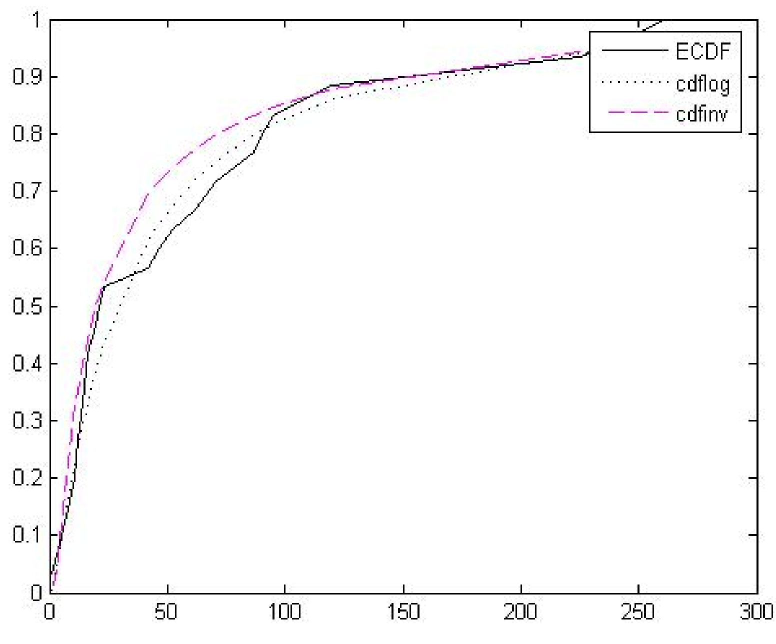

Figure 5: The CDF for both distributions and the ECDF (Kolmogrove- Smirnove CDF) for data set (3)

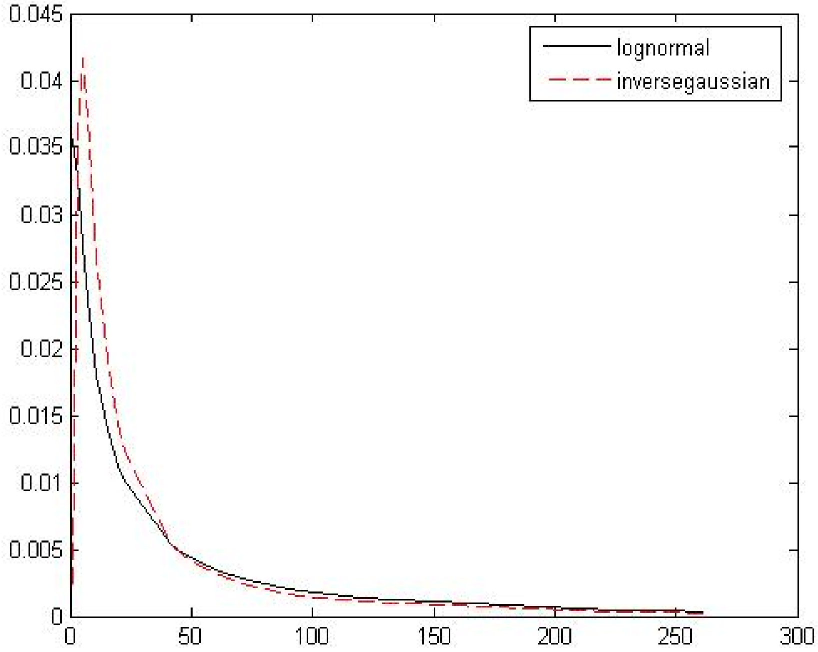

Figure 6: The p.d.f for both distributions for data set (3)

5.1.4 Data Set (4)

Gacula and Kubala (1975) give the following data on shelf life (days) of a food product:
24,24,26,26,32,32,33,33,33,35,41,42,43,47,48,48,48,50,52,54,55, 57,57,57,57,61

Table (4): The m.l.e for both distribution parameters and Kolmogrove- Smirnove (K-S) statistic

IG	LOGN
$\hat{\mu} = \mathbf{42.88}$	$\hat{\theta} = \mathbf{3.718}$
$\hat{\lambda} = \mathbf{484.2519}$	$\hat{\sigma} = \mathbf{0.2924}$
K-S = 0.1378	K-S = 0.1359

Again, these data belong to the both distributions, but $\ln L$ = 0.072 > 0. That is, the IG distribution is reasonable for them.

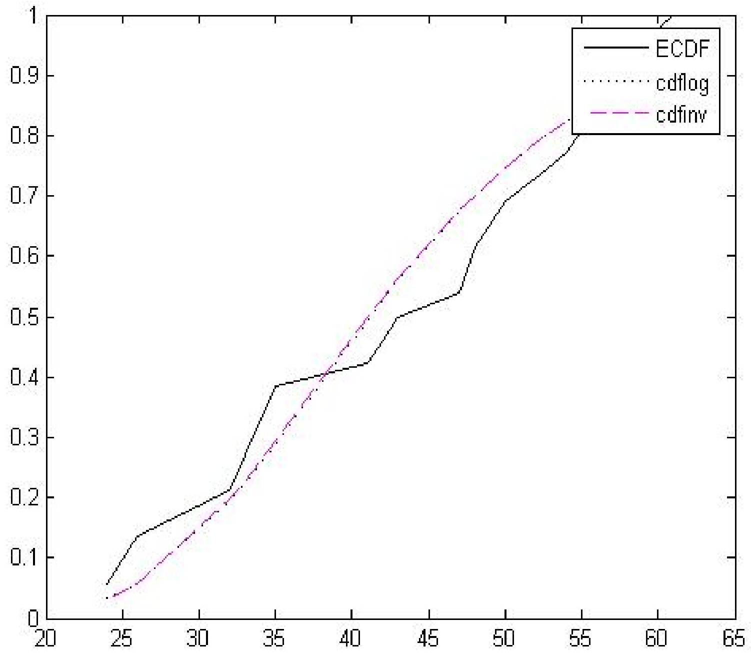

Figure 7: The CDF for both distributions and the ECDF (Kolmogrove- Smirnove CDF) for data set (4)

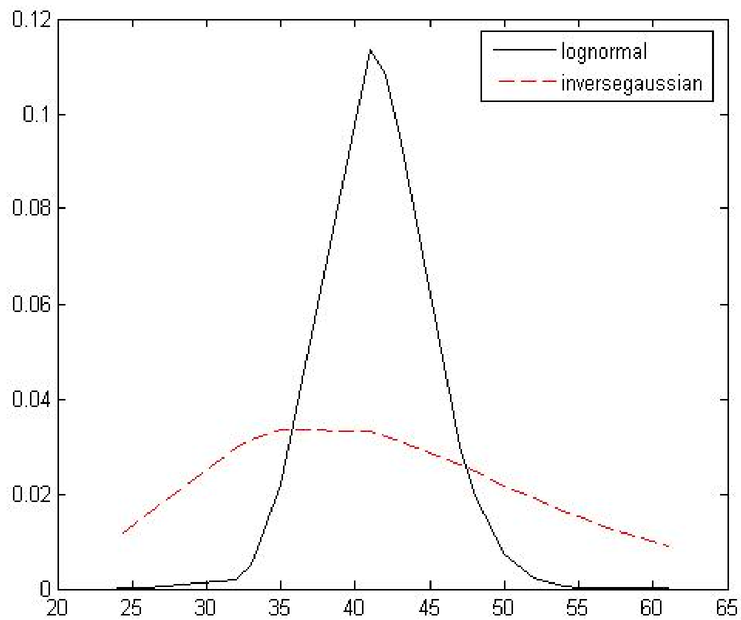

Figure 8: The p.d.f for both distributions for data set (4).

5.1.5 Data Set (5)

Ang and Tang (1975) use fracture toughnesses of MIG welds
54.4,62.6,63.2,67,70.2,70.5,70.6,71.4,71.8,74.1,74.1,74.3,78.8,81.8,83,84.4,85.3,86.9,87.3

Table (5): The m.l.e for both distribution parameters and Kolmogrove- Smirnove (K-S) statistic

IG	LOGN
$\hat{\mu} = \mathbf{74.3}$	$\hat{\theta} = \mathbf{4.3008}$
$\hat{\lambda} = \mathbf{4924.07}$	$\hat{\sigma} = \mathbf{0.1224}$
K-S = 0.133	K-S = 0.132

The value of $\ln L$ is -0.0012 < 0. It suggest that the LOGN distribution to be preferred over the IG distribution. According to the K-S test these data belong to both distributions.

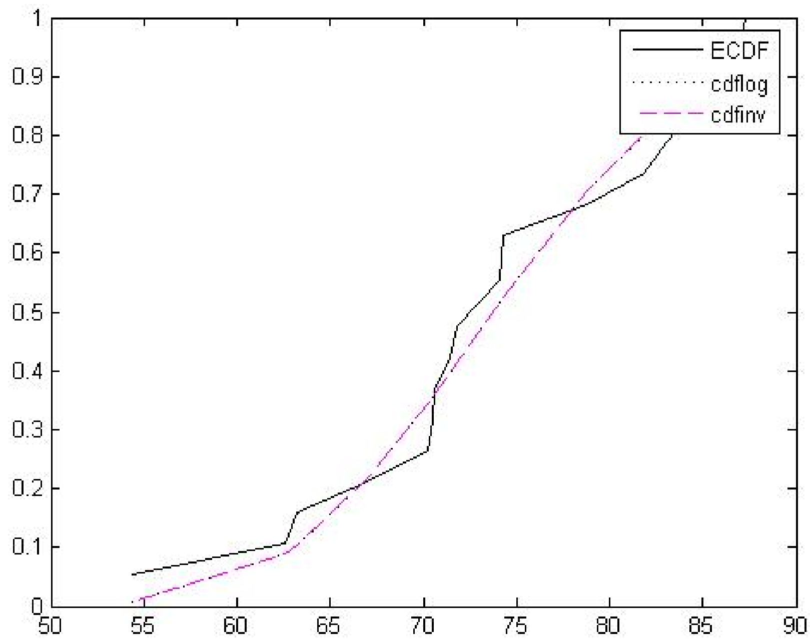

Figure 9: The CDF for both distributions and the ECDF (Kolmogrove- Smirnove CDF) for data set (5).

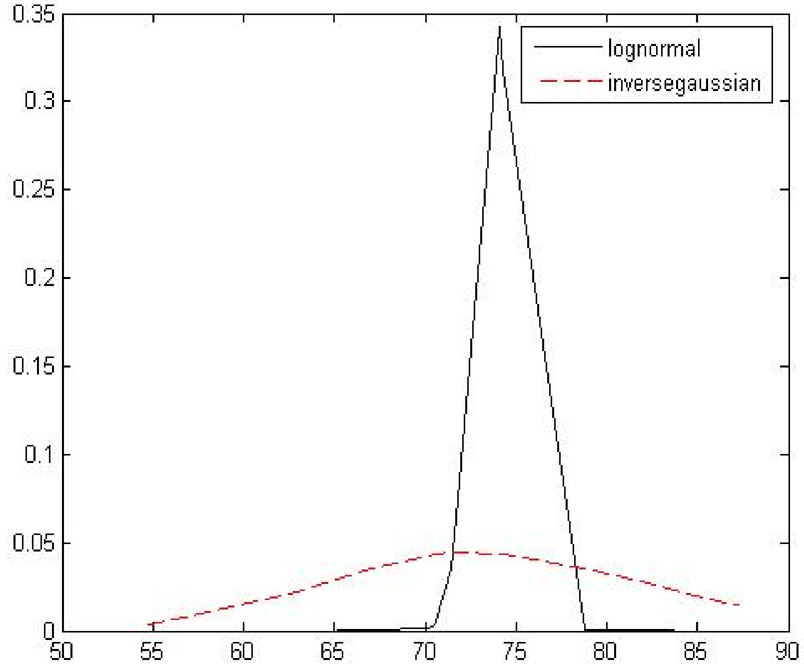

Figure 10: The p.d.f for both distributions for data set (5).

5.1.6 Data Set (6)

The sixth set gives data of precipitation (inches) from Jug Bridge, Maryland (Chhikara and Folks, 1978).
1.01,1.11,1.13,1.15,1.16,1.17,1.17,1.2,1.52,1.54,1.54,1.57,1.64,1.73,1.79,2.09,2.09,2.57,2.75,2.93,3.19,3.54,3.57,5.11,5.62

Table (6): The m.l.e for both distribution parameters and Kolmogrove- Smirnove (K-S) statistic

IG	LOGN
$\hat{\mu} = 2.1556$	$\hat{\theta} = 0.6375$
$\hat{\lambda} = 8.082$	$\hat{\sigma} = 0.4893$
K-S = 0.15	K-S = 0.145

Because of the value of $\ln L$ = 0.2815 > 0, we conclude that the data well-fitted by the IG distribution.

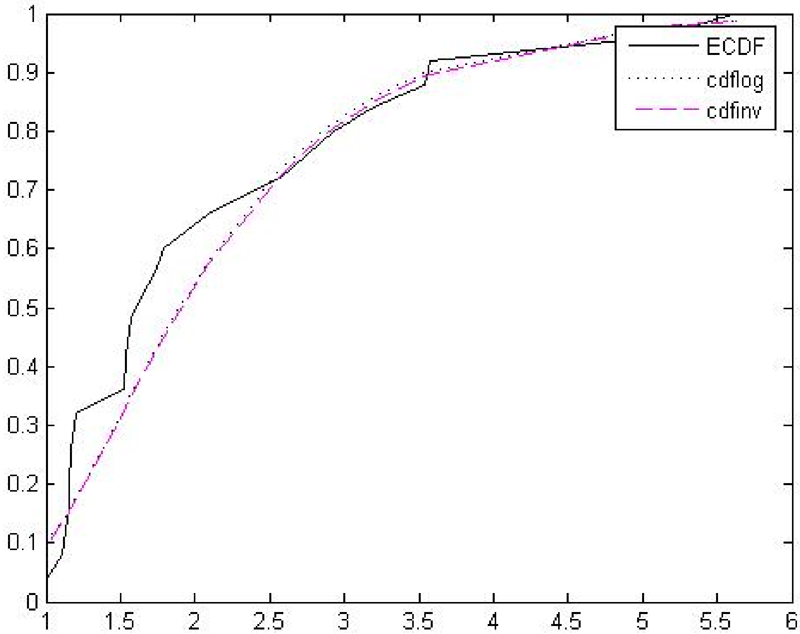

Figure 11: The CDF for both distributions and the ECDF (Kolmogrove- Smirnove CDF) for data set (6).

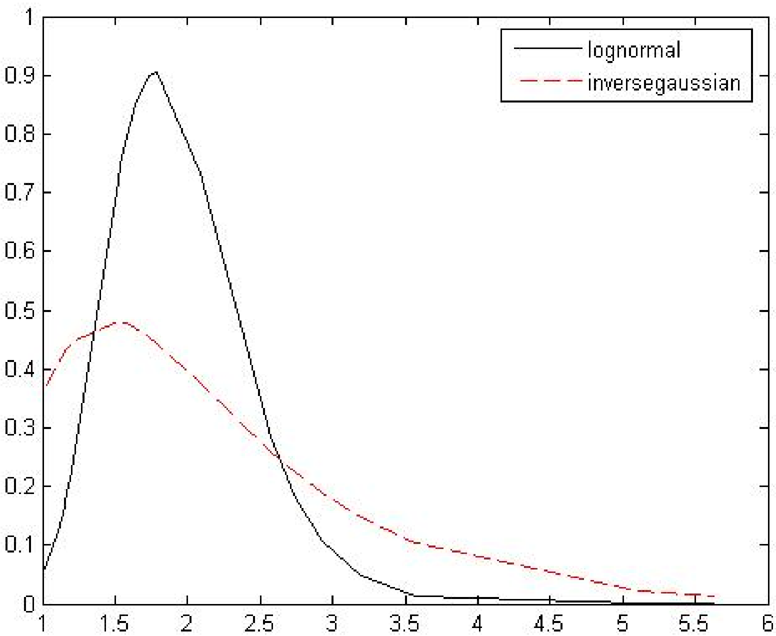

Figure 12: The p.d.f for both distributions for data set (6).

5.1.7 Data set (7)

Here runoff amounts at Jug Bridge, Maryland are given (Chhikara and Folks, 1978).

0.17,0.19,0.23,0.33,0.39,0.39,0.4,0.45,0.52,0.56,0.59,0.64,0.66,0.7,0.76,0.77,0.78,0.95,0.97,1.02,1.12,1.24,1.59,1.74,2.92

Table (7): The m.l.e for both distribution parameters and Kolmogrove- Smirnove (K-S) statistic

IG	LOGN
$\hat{\mu} = \mathbf{0.8032}$	$\hat{\theta} = -\mathbf{0.4407}$
$\hat{\lambda} = \mathbf{1.4397}$	$\hat{\sigma} = \mathbf{0.6682}$
K-S = 0.07	K-S = 0.0668

According to the values of K-S test of the two distributions, we conclude that the data are very well described by these two distributions. But $\ln L$ = -0.0153 < 0, we prefer that the LOGN distribution will be more reasonable.

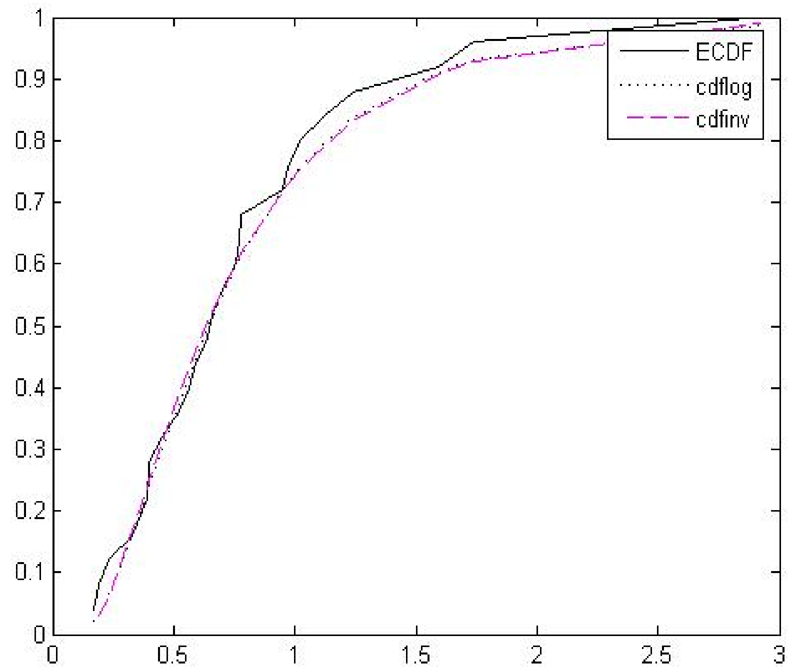

Figure 13: The CDF for both distributions and the ECDF (Kolmogrove- Smirnove CDF) for data set (7).

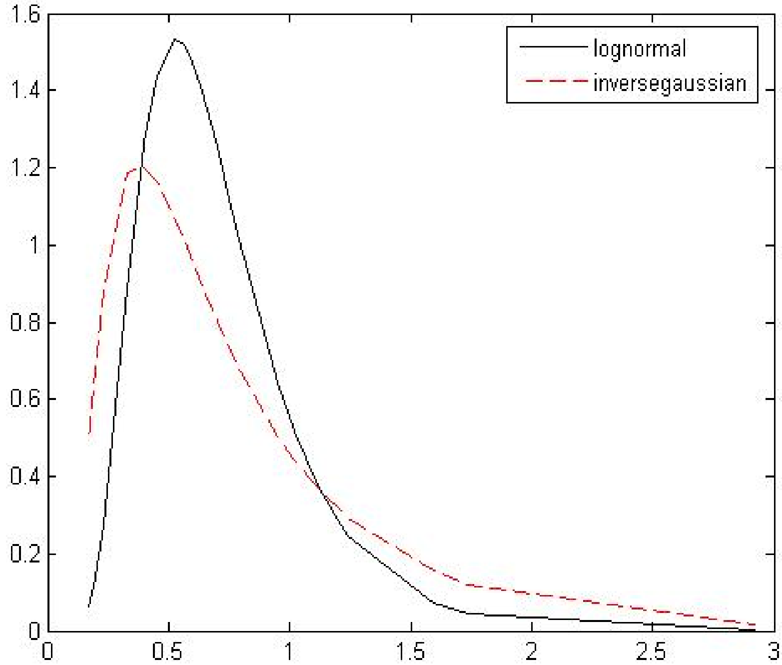

Figure 14: The p.d.f for both distributions for data set (7).

5.1.8 Data set (8)

Kumagai et al (1989) presented the following time series data for toluene exposure concentrations (8 hr TWAs) for a worker doing stain removing.
0.9,1.1,1.9,2.1,2.6,2.9,3.1,3.2,4.9,4.9,5.2,5.8,6.2,6.9,7.8,8.3,8.7,10.5,11.1,13.6,16.6,17.4,20.4,21.9,22.4,50.9,57.4,58.3,58.6,66.9

Table (8): The m.l.e for both distribution parameters and Kolmogrove- Smirnove (K-S) statistic

IG	LOGN
$\hat{\mu} = 16.75$	$\hat{\theta} = 2.1643$
$\hat{\lambda} = 6.4641$	$\hat{\sigma} = 1.1765$
K-S = 0.0952	K-S = 0.099

According to the values of K-S test of the two distributions, we conclude that the data are very well described by these two distributions. But $\ln L$ = 0.406 > 0, we prefer that the IG distribution will be more reasonable.

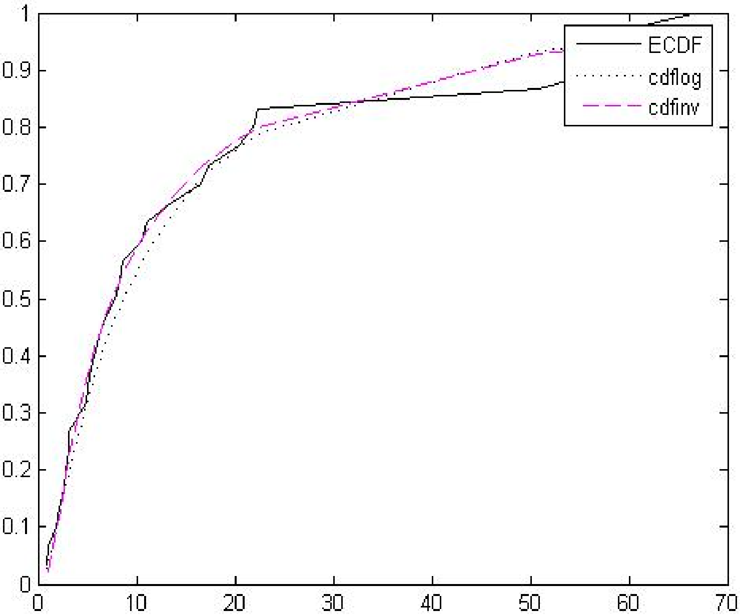

Figure 15: The CDF for both distributions and the ECDF (Kolmogrove- Smirnove CDF) for data set (8).

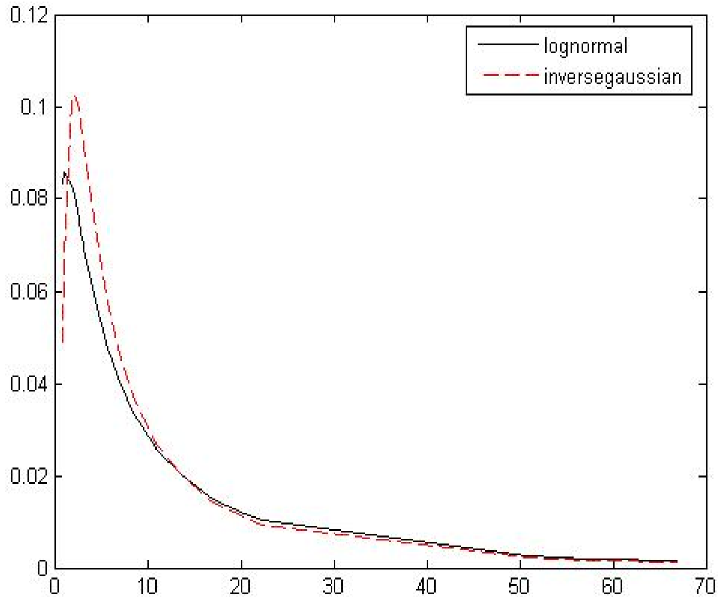

Figure 16: The p.d.f for both distributions for data set (8).

5.1.9 Data Set (9)

Kumagai and Matsunaga (1995) give these data
1.5,1.7,2.1,2.2,2.4,2.5,2.6,3.8,3.8,4.2,4.3,5.6,6,7,7.5,9.3,9.9,10.2,
10.6,12.3,12.9,13.7,14.1,17.8,27.6,31,42,45.6,51.9,91.3,131.8

Table (9): The m.l.e for both distribution parameters and Kolmogrove- Smirnove (K-S) statistic

IG	LOGN
$\hat{\mu} = 19.0065$	$\hat{\theta} = 2.20393$
$\hat{\lambda} = 7.2326$	$\hat{\sigma} = 1.1733$
K-S = 0.088	K-S = 0.095

The value of $\ln L$ is 1.4611 > 0. It suggest that the IG distribution to be preferred over the LOGN distribution. According to the K-S test these data belong to both distributions.

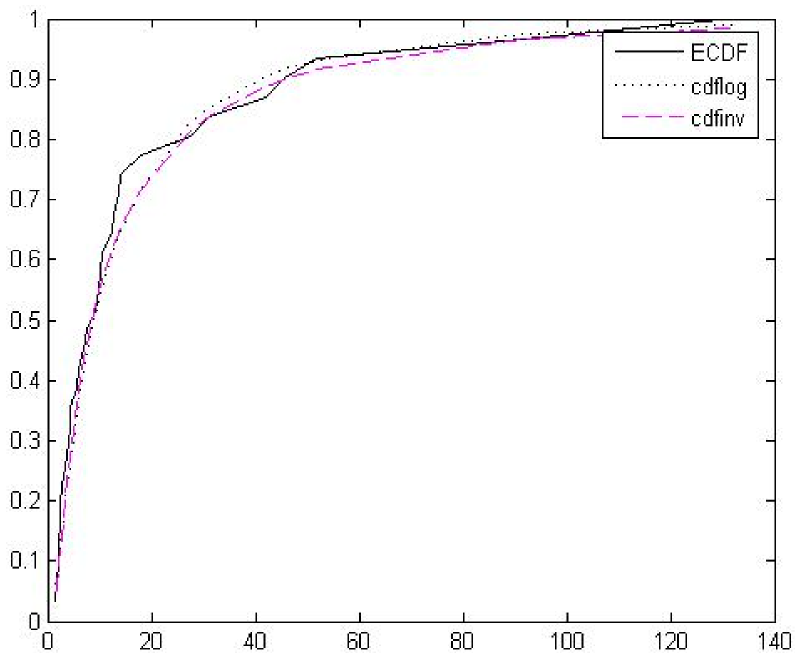

Figure 17: The CDF for both distributions and the ECDF (Kolmogrove- Smirnove CDF) for data set (9).

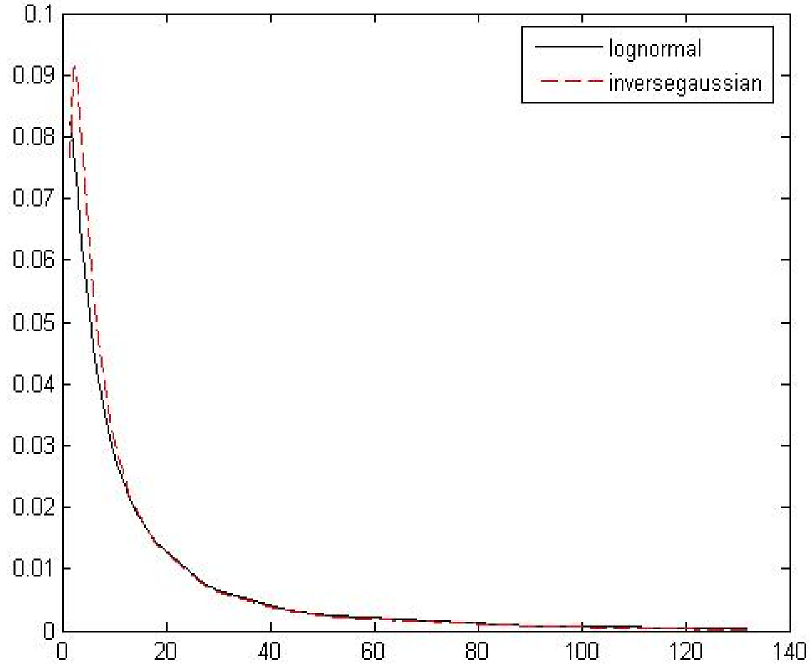

Figure 18: The p.d.f for both distributions for data set (9).

5-2 Simulated Data Analysis

5.2.1 Data Set (10)

This data set represents the LOGN with $\hat{\theta}=0.5$ and $\hat{\sigma}=1.5$, the data are: 0.2963,0.4447,0.483,0.5819,0.8603,0.9078,0.9095,1.0099,1.1677,1.4404,1.4976,1.5451,1.6825,1.7319,2.0701,2.4695,2.6095,3.2994,3.3531,3.498.

Table (10): The m.l.e for both distribution parameters and Kolmogrove- Smirnove (K-S) statistic

IG	LOGN
$\hat{\mu}=1.59$	$\hat{\theta}=0.252$
$\hat{\lambda}=2.7$	$\hat{\sigma}=0.6903$
K-S = 0.0833	K-S = 0.07489

According to the values of K-S test of the two distributions, we conclude that the data are very well described by these two distributions. But $\ln L$ = -0.00236 < 0, we prefer that the LOGN distribution will be more reasonable.

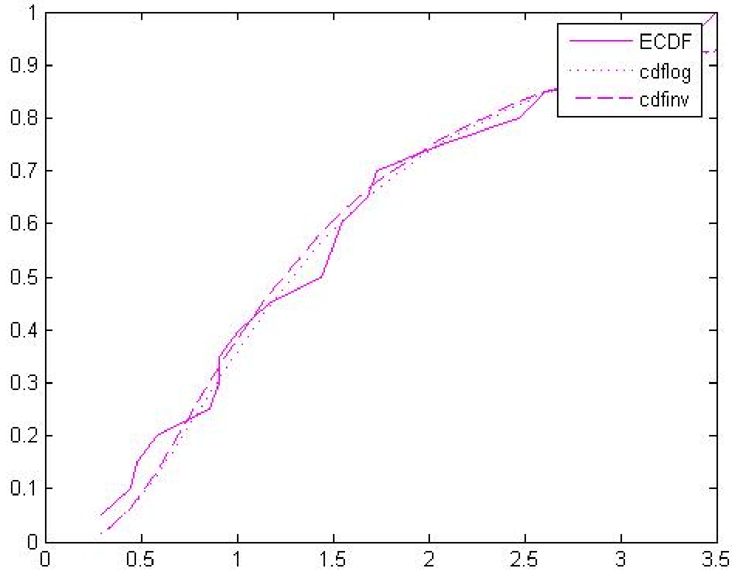

Figure 19: The CDF for both distributions and the ECDF (Kolmogrove- Smirnove CDF) for data set (10).

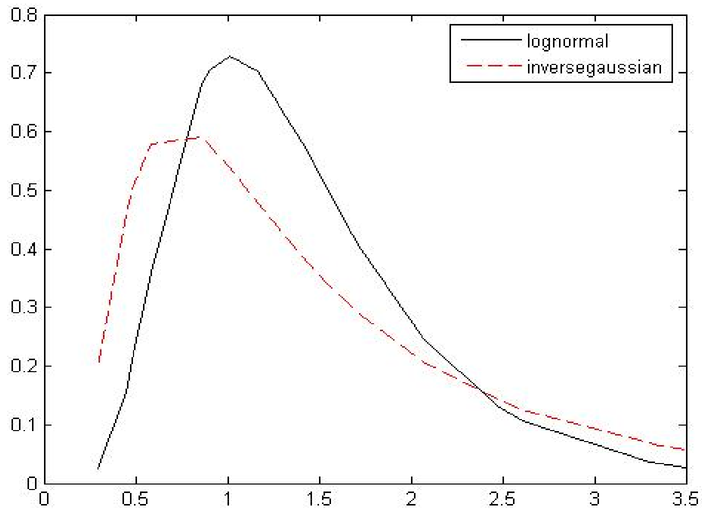

Figure 20: The p.d.f for both distributions for data set (10).

5.2.2 Data Set (11)

This data set represents the IG with $\hat{\mu}=75$ and $\hat{\lambda}=290$, the data are: 26.6183,27.635,29.9275,31.609,34.8973,41.1112,44.2393,46.6927,50.645,51.5158,57.5755,59.1067,60.3766,62.2319,65.4591,67.3522,67.754,69.5194,73.7422,74.0017,75.2821,85.4949,90.9635,92.0092,92.1252,92.6779,97.1245,99.2954,110.7208,118.0211,118.1289,122.0813,124.4119,148.0396,198.2293.

Table (11): The m.l.e for both distribution parameters and Kolmogrove- Smirnove (K-S) statistic

IG	LOGN
$\hat{\mu}=77.33$	$\hat{\theta}=4.2352$
$\hat{\lambda}=295.406$	$\hat{\sigma}=0.4845$
K-S = 0.0625	K-S = 0.06347

Both K-S values are significant, but the value of $\ln L$ is 0.07 > 0, therefore the IG distribution is more suitable than LOGN distribution.

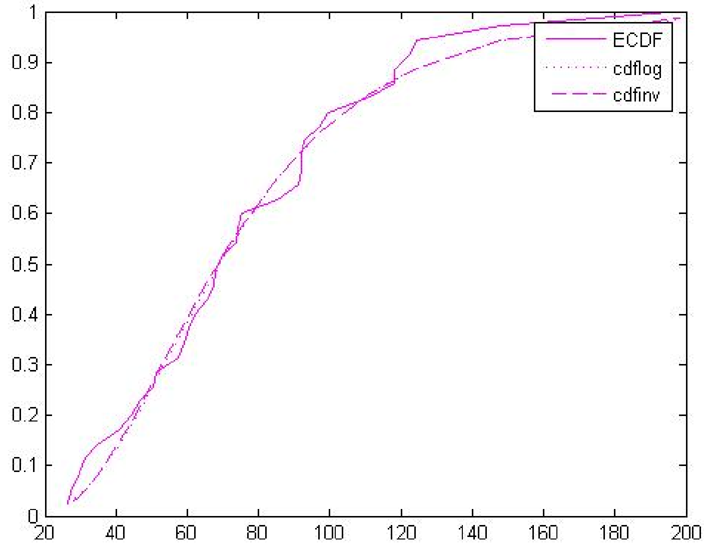

Figure 21: The CDF for both distributions and the ECDF (Kolmogrove- Smirnove CDF) for data set (11).

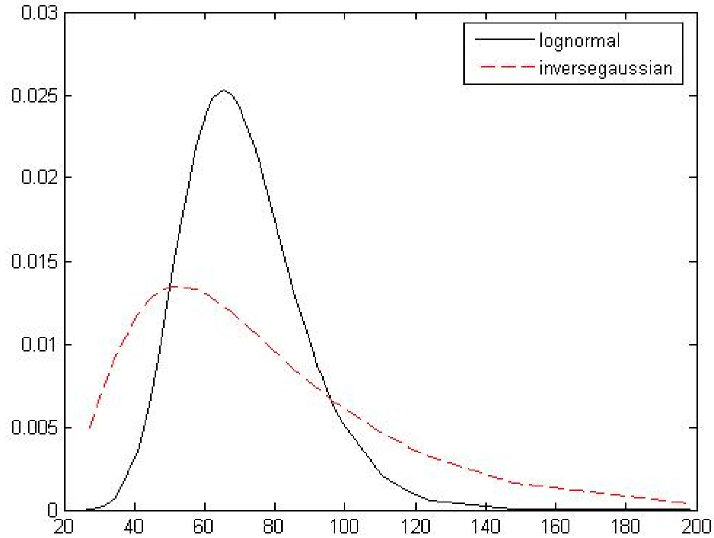

Figure 22: The p.d.f for both distributions for data set (11).

5.2.3 Data Set (12)

This data set represents the LOGN with $\hat{\theta}=3.5$ and $\hat{\sigma}=1.5$,the data are: 1.2763,2.9858,5.4538,6.7889,7.2721,7.2887,8.3074,9.8983,12.6115,26.1781,30.3021,30.8047,32.1469,32.1603,46.0156,48.7135,58.5866,70.9242,73.1941,80.3927,83.2372,90.2786,197.6024,276.6387,419.3244.

Table (12): The m.l.e for both distribution parameters and Kolmogrove- Smirnove (K-S) statistic

IG	LOGN
$\hat{\mu}=66.33$	$\hat{\theta}=3.3403$
$\hat{\lambda}=12.839$	$\hat{\sigma}=1.4008$
K-S = 0.1379	K-S = 0.09281

Both K-S values are significant, but the value of $\ln L$ is -1.4218 > 0, therefore the LOGN distribution is more suitable than IG distribution**.**

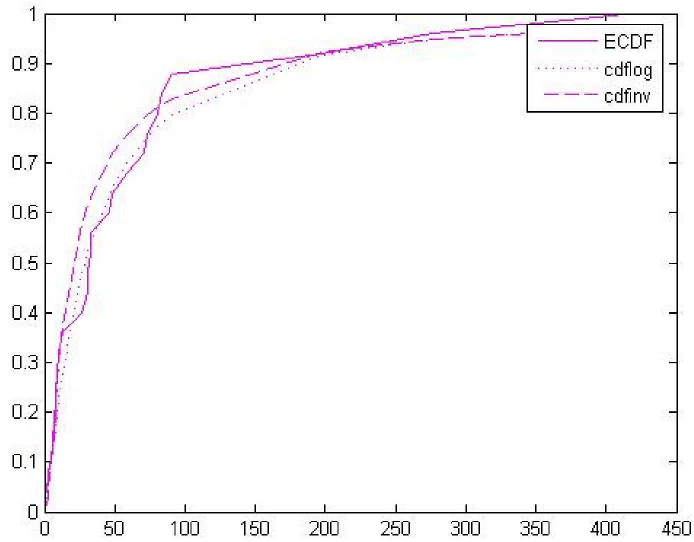

Figure 23: The CDF for both distributions and the ECDF (Kolmogrove- Smirnove CDF) for data set (12).

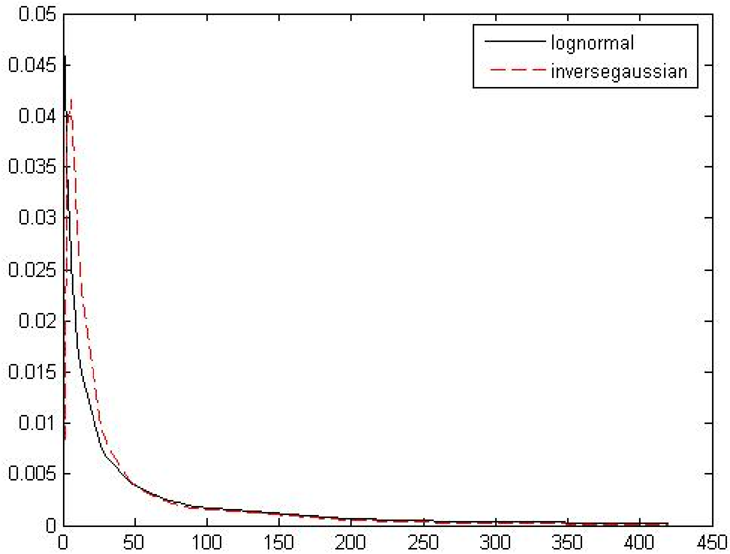

Figure 24: The p.d.f for both distributions for data set (12).

6- Conclusions

1- Through tables (1), (4), (6), (8),(9), and (11) we see that these data have the same distributions according to the value of K-S test but the value of $\ln L$ suggests that these data to have the IG distribution rather than the LOGN distribution.

2- From tables (2), (3), (5),(7),(10), and (12) the data have both distributions, but according to the value of $\ln L$ the LOGN distribution is more suitable than IG distribution.

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