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Exponentiated Exponential Distribution as a Failure Time Distribution

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ABSTRACT

Two machines failure time data and two simulated examples have been studied through their reliability function and failure rate using the exponentiated exponential distribution. It could be considered as a well positively skewed distribution, since it has increasing, decreasing, and constant rate function depending on the shape parameter only, and it has an explicit expression of the distribution and reliability functions.

Keywords: Failure time distribution, Exponentiated exponential distribution, rate function, reliability function.

التوزيع الآسي المرفوع أسيا بوصفه توزيعا لوقت الفشل

الملخص

تم استخدام التوزيع الآسي المرفوع أسيا بوصفه توزيعا لوقت الفشل من خلال دراسة دالة المعولية ومعدل الفشل لمجموعتين من بيانات أوقات الفشل – بيانات حقيقية وبيانات مولدة . يمكن ان يعد هذا التوزيع توزيعا ملائما للبيانات ذات الالتواء الموجب . إن معدل الفشل في هذا التوزيع يكون متزايدا، متناقصا، وثابتاً بالاعتماد على قيمة معلمة شكل التوزيع فقط. كذلك إن لهذا التوزيع دالة احتمال تراكمية ودالة معولية واضحة وبسيطة .

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1-Introduction

Failure time data (survival time data) measure the time to a certain event, such as failure, death. These times are subject to be random variables. These data have the property that are non-negative and typically have skewed distribution (Lee, 2003). Many skewed distributions like lognormal, gamma, and weibull are used as a failure time distributions (lifetime distributions) (Kalbfleisch and Prentice, 2002).

The choice of distribution is often made on the basis of how well the data appear to be fitted by the distribution. Recently, a new skewed failure time distribution named exponentiated exponential distribution (or generalized exponential distribution) has been introduced and studied by Gupta and Kundu (1999, 2001, 2003, 2004, 2007). They observed that this distribution can be used in place of gamma and weibull distributions, since the two parameters of the gamma, Weibull, and exponentiated exponential distributions have increasing as well as decreasing hazard function depending on the value of the shape parameter, also they have a constant hazard function when the shape parameter is equal to one. (Gupta and Kundu, 1999).

The main disadvantage of the gamma distribution is the distribution function or the reliability function cannot be expressed in a closed form if the shape parameter is not an integer.(Gupta & Kundu, 2001).

The probability density function of an exponentiated exponential distribution (EE) with scale (λ) and shape (α) parameters is

$$f(x;\alpha,\lambda)=\begin{cases}\dfrac{\alpha\lambda e^{-\lambda x}}{(1-e^{-\lambda x})^{-(\alpha-1)}} & , \quad x,\alpha,\lambda>0 \\ 0 & \textit{elsewhere}\end{cases} \quad ..\ (1)$$

The p.d.f is an unimodal density. It is log-convex if $\alpha \le 1$ and log-concave if $\alpha > 1$.

The CDF is

$$F(x)=\int_0^x f(x;\alpha,\lambda)\,dt$$

$$=(1-e^{-\lambda x})^{\alpha} \qquad , \quad x,\alpha,\lambda>0 \qquad .....\ (2)$$

The moment generating function $M(t)$, the mean, and the variance are given respectively,

$$M_x{}^{(t)}=\Gamma(\alpha+1)\Gamma\left(1-\frac{t}{\lambda}\right)/\Gamma\left(\alpha-\frac{t}{\lambda}+1\right) \qquad ......\ (3)$$

$$\mu_x=\frac{1}{\lambda}(\Psi(\alpha+1)-\Psi(1)) \qquad ......\ (4)$$

$$\sigma_x{}^2=\frac{1}{\lambda^2}(\Psi'(1)-\Psi'(\alpha+1)) \qquad ......\ (5)$$

Where $\Psi(\alpha)=\ln\Gamma\alpha$ is called the digamma function and $\Psi'(\alpha)=\dfrac{d\ln\Gamma\alpha}{d\alpha}$ its derivative, (Gupta and Kundu, 1999).

The median is

$$me_x = -\frac{\ln(1-(0.5)^{1/\alpha})}{\lambda} \qquad .....(6)$$

2- Reliability Function and Failure Rate

The reliability function, $R(t)$, is the probability of no failure until time t,

$$\begin{aligned} R(t) &= P(T>t) \\ &= 1-P(T\le t) \\ &= 1-F(t) \end{aligned}$$

So,

$$R(t)=1-(1-e^{-\lambda t})^{\alpha} \qquad , \quad t,\alpha,\lambda>0 \qquad .....(7)$$

The failure rate (Hazard Function), $r(t)$, is defined as:

$$r(t)=\frac{f(t)}{R(t)}, \; t>0$$

Thus the $r(t)$ of an exponentiated exponential distribution with parameters α and λ is

$$r(t)=\frac{\alpha\lambda e^{-\lambda t}}{(1-e^{-\lambda t})(1-(1-e^{-\lambda t})^{-\alpha})} \quad , \quad t,\alpha,\lambda>0 \quad ....(8)$$

Several typical failure rate curves are given in figure 1. Inspection of these curves makes it obvious that the $r(t)$ be increasing when $\alpha > 1$,decreasing when $\alpha < 1$,and constant when $\alpha = 1$.

$r(t)$

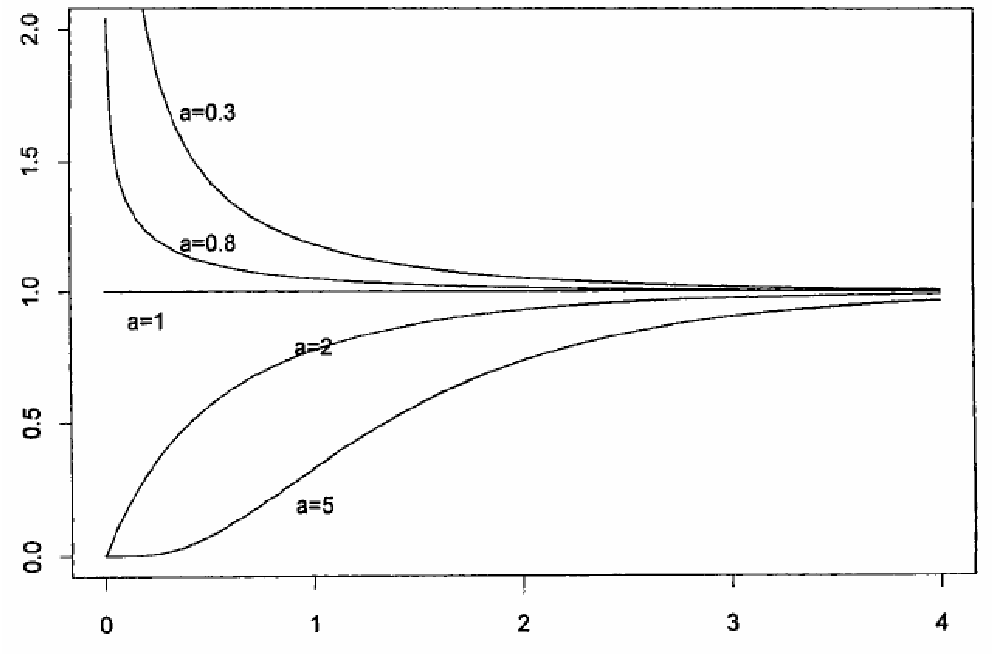

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Figure (1): shows several failure rate curves.

3- Mean Time to Failure and Mean Residual Lifetime

The mean time to failure (MTTF) is defined as the expectation of t , i.e.

$$MTTF = E(t) = \int_0^\infty t\, f(t)dt \qquad \text{………..(9)}$$

So,

$$MTTF = \frac{1}{\lambda}(\Psi(\alpha+1) - \Psi(1)) \qquad \text{………………(10)}$$

The mean residual lifetime (μ_T) is of interest of many fields such as reliability, survival analysis and it is defined as the follows: Let S be the residual lifetime consisting of the period from the T until the time of failure given that there was no failure prior to T, then the μ_T is

$$\mu_T = \frac{1}{R(t)} \int_T^{\infty} (t - T)\, f(t)dt \qquad \text{………………….(11)}$$

So,

$$\mu_T = \frac{\int_T^{\infty} t\, f(t)dt \; - \; T\int_T^{\infty} f(t)dt}{1 - (1 - e^{-\lambda T})^{\alpha}} \qquad \text{……………………(12)}$$

4- Point Estimation of Parameters

Let $x_1, x_2, \ldots\ldots, x_n$ be a random sample from (1), then the likelihood function and the natural logarithm of the likelihood function are

$$L(\alpha,\lambda) = (\alpha\lambda)^n e^{-\lambda \sum_i x_i} \prod_{i=1}^{n} (1 - e^{-\lambda x_i})^{\alpha-1} \qquad \text{……………(13)}$$

$$L = \ln L(\alpha,\lambda) = n\ln\alpha + n\ln\lambda + (\alpha-1)\sum_{i=1}^{n} \ln(1 - e^{-\lambda x_i}) - \lambda \sum_{i=1}^{n} x_i \qquad \text{..(14)}$$

By using the maximum likelihood method, one can find the point estimator for α and λ, so

$$\frac{\partial L}{\partial \alpha} = n\alpha^{-1} + \sum_{i=1}^{n} \ln(1 - e^{-\lambda x_i}) = 0 \qquad \text{...............(15)}$$

$$\frac{\partial L}{\partial \lambda} = n\lambda^{-1} + (\alpha - 1)\sum_{i=1}^{n} \frac{x_i\, e^{-\lambda x_i}}{1 - e^{-\lambda x_i}} - \sum_{i=1}^{n} x_i = 0 \qquad \text{...........(16)}$$

From (14) we obtain the m.l.e of α as a function of λ

$$\hat{\alpha}(\lambda) = -\frac{n}{\sum_{i=1}^{n} \ln(1 - e^{-\lambda t_i})} \qquad \text{............(17)}$$

Substitute (17) in (15) to get

$$L(\lambda) = Const. + n \ln \lambda - \sum_{i=1}^{n} \ln(1 - e^{-\lambda x_i}) - \lambda \sum_{i=1}^{n} x_i - n \ln\left[\left(\sum_{i=1}^{n} \ln(1 - e^{-\lambda x_i}) \right) \right] . \text{(18)}$$

And using a simple iterative method to find the m.l.e of λ.

The m.l.e of the reliability function $R(t)$ and the failure rate $r(t)$ is now obtained by replacing the m.l. estimators of $\hat{\alpha}$ and $\hat{\lambda}$. Accordingly, the m.l.e of $R(t)$ and $r(t)$ are

$$\hat{R}(t) = 1 - (1 - e^{-\hat{\lambda} t})^{\hat{\alpha}} \qquad \text{..................(19)}$$

$$\hat{r}(t) = \frac{\hat{\alpha}\hat{\lambda} e^{-\hat{\lambda} t}}{(1 - e^{-\hat{\lambda} t})(1 - (1 - e^{-\hat{\lambda} t})^{-\hat{\alpha}})} \qquad \text{..................(20)}$$

5- Case Study

The aim is to estimate the reliability function and the failure rate. This case study is divided into two parts: First for real data and, second for simulated data.

5-1: Real Data

Here we study two machines from Babel tyiers factory (Iraq). The data was obtained from (Al-Kazrajy, 2001).These data represent the failure time in hours.

5-1-1: Machine (1): Cutting Layers Machine

The data represent the failure time in hours. They are [1.00,1.00,5.00,5.50,12.50,16.75,17.75,20.75,22.50,22.75,25.00, 25.00,27.25,30.25,43.75,45.00,48.00,48.25,97.50,99.75,136.75,1 43.50,207.75,215.00, 225.50,235.00,283.50,567.00,970.50]

5-1-2: Machine (2): Coating I Machine

These data represent the failure time in hours. [3.5,6.5,10.5,23.25,23.5,43.5,69,70.5,75.5,83.25,95.5,109.5,111. 25,144,164,167.25,253,383.75,417.75,428.25,453,1215]

The goodness of fit tests, Kolmogrove- Smirnove (K-S) and Anderson-Darling (A^2), show that these data follows the EE distribution. Table (1) shows the values of the tests.

Table (1): The Values of K-S and A^2 tests when α=0.05[2]

	Machine (1)	Machine (2)
K-S	0.1906 (0.252)	0.1317 (0.2899)
A^2	0.728 (2.501)	0.314 (2.5)

The m.l.e for the α and λ for the two machines are shown in table (2).

[2] The values between brackets are the critical values of the tests, which suggest the data belong to the exponentiated exponential distribution.

Table (2) : The m.l.e for α and λ

	Machine (1)	Machine (2)
$\hat{\alpha}$	0.5731	0.7396
$\hat{\lambda}$	0.0054	0.0041

Figures (2) and (3) show the results that we obtained from the equations (19) and (20).

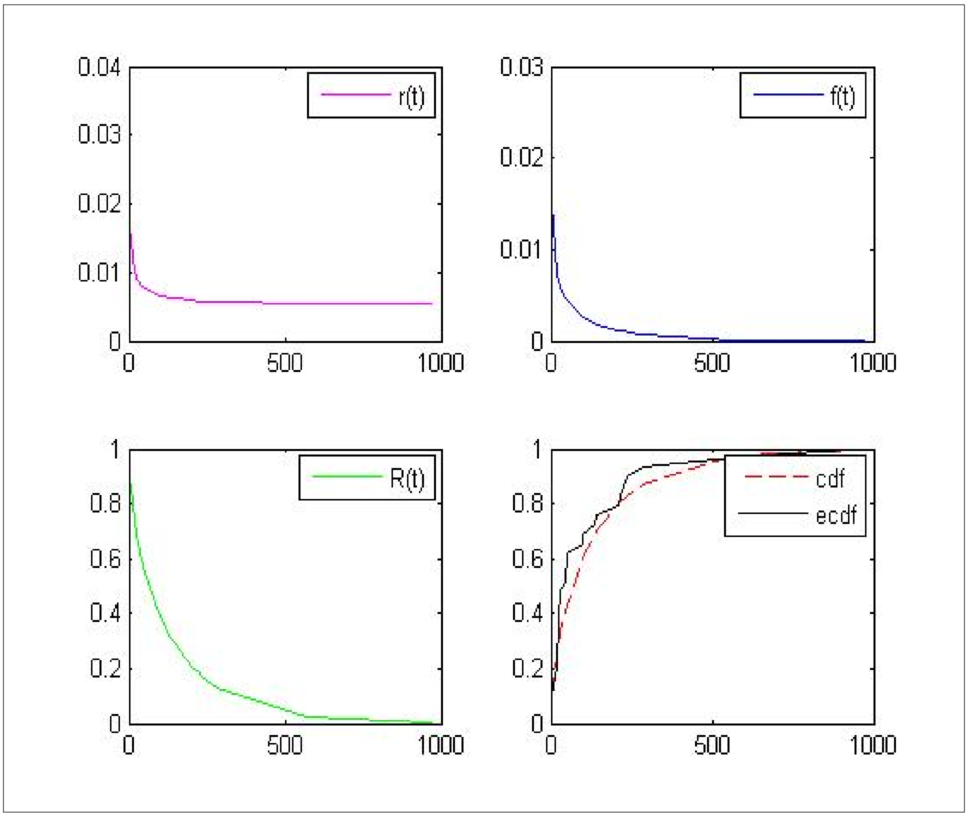

Figure (2): $f(t), F(t), R(t),$ and $r(t)$ of machine (1).

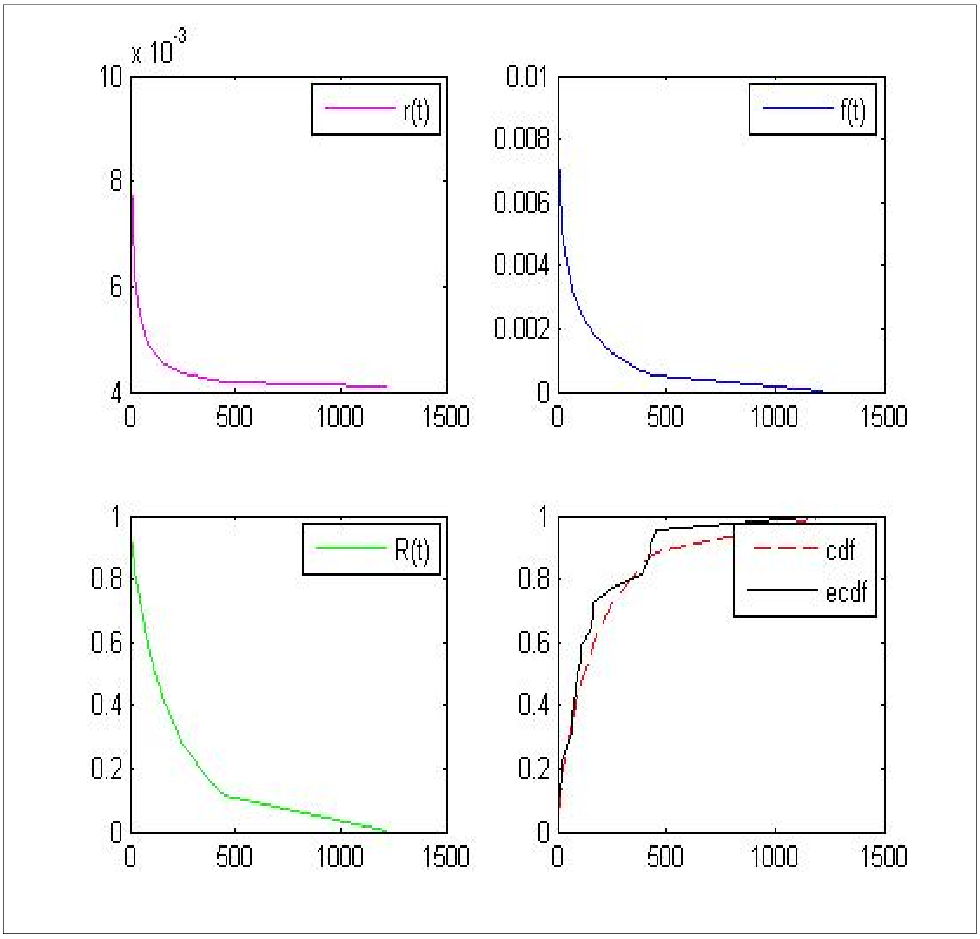

Figure (3): $f(t), F(t), R(t),$ and $r(t)$ of machine (2)

5-2: Simulated Data

Here we give two examples. The generation of exponentiated exponential distribution is very simple. Let $U \sim Uni(0,1)$, then $x = (-\ln(1 - U^{1/\alpha})/\lambda)$ follows $EE(\alpha, \lambda)$.

5-2-1: Example (1):

Here we set $\alpha = 2.5$, $\lambda = 0.5$,and $n = 100$ to simulated the data. The m.l.e. of α and λ are 3.689 and 0.6057 respectively.

Figure (4) shows the results that we obtained from the equations (19) and (20).

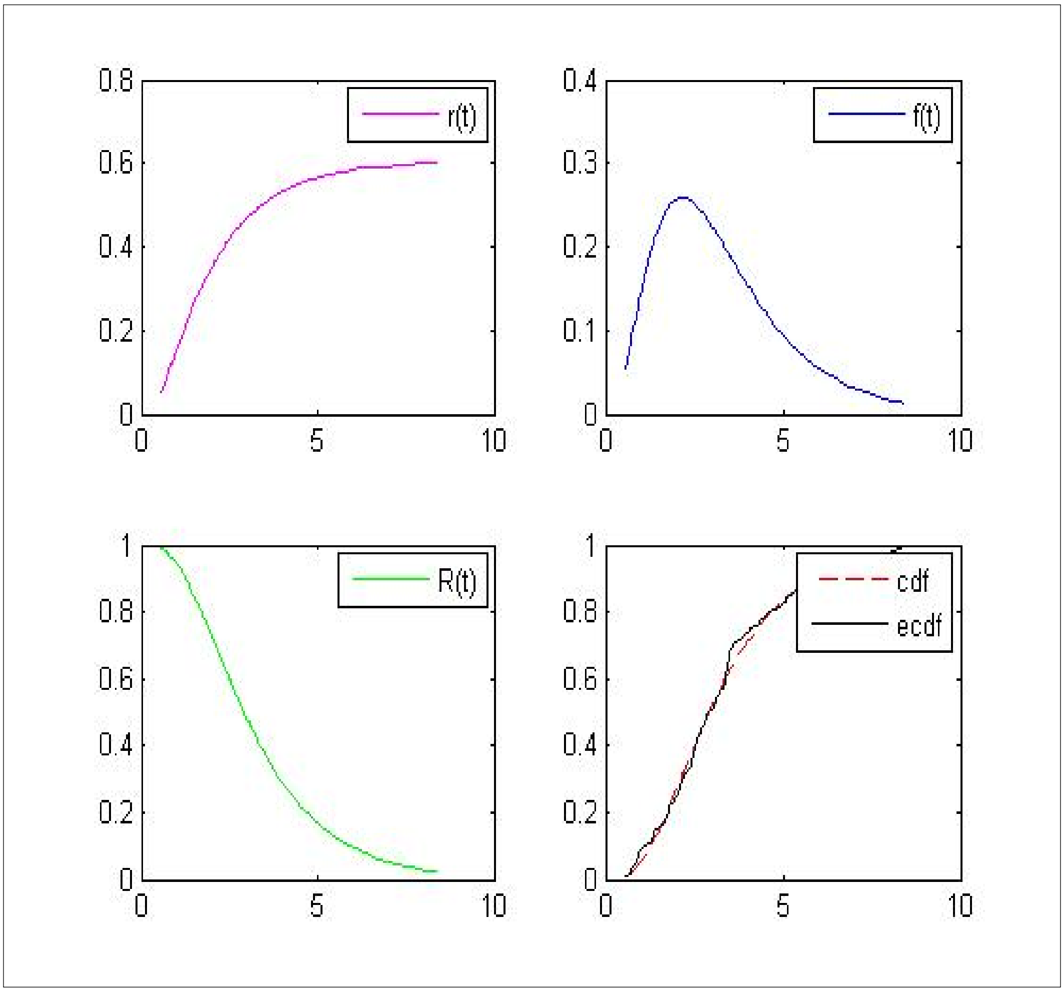

Figure (4): $f(t), F(t), R(t),$ and $r(t)$ of example (1).

5-2-2: Example (2):

Here $\alpha = 1$, $\lambda = 0.5$, and $n = 20$ to simulated the data. The m.l.e. of α and λ are $\hat{\alpha} = 3.689$ and $\hat{\lambda} = 0.6057$ respectively. The results that we obtained from the equations (19) and (20) are showed in figure (5).

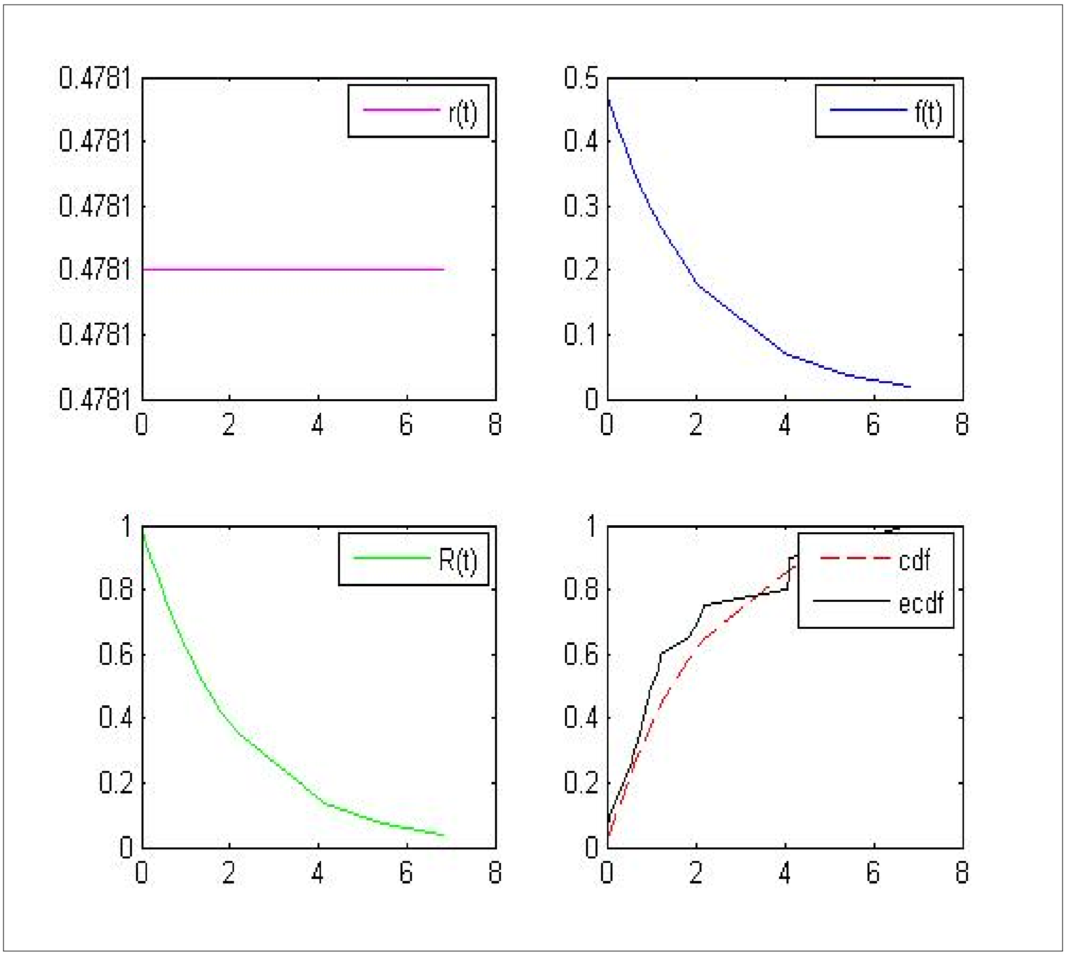

Figure (5): $f(t), F(t), R(t),$ and $r(t)$ of example (2).

4-Conclusions

In this paper we consider the exponentiated exponential distribution as a failure time distribution. As we see from figures (2) and (5) the rate function has decreasing, increasing, and constant form when $\alpha < 1$, $\alpha > 1$, and $\alpha = 1$ respectively. The probability density function are unimodal and have a log-convex form in figure (1),(2) and (4) because $\alpha \le 1$,and a log-concave in figure (3) since $\alpha > 1$.

5-References

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